## Eugène-Maurice-Pierre Cosserat (1866 – 1931)

From Wikipedia, the free encyclopedia,
http://en.wikipedia.org/wiki/Eugène_Cosserat

Eugène-Maurice-Pierre Cosserat was a French mathematician and astronomer. Born in Amiens, he studied at the École Normale Supérieure from 1883 to 1888. He was on Science faculty of Toulouse University from 1889 and director of its observatory from 1908, a position he held for the rest of his life. He was elected to the Académie des Sciences in 1919. His studies included the rings and satellites of Saturn, comets and double stars, but is best remembered for work with his engineer brother François on surface mechanics, particularly problems of elasticity.

## Cosserat Theory

by Prof. Dr. Patrizio Neff, Lehrstuhl fur Nichtlineare Analysis, Universitat Duisburg-Essen
from http://www.uni-due.de/mathematik/ag_neff/cosserat

### The Cosserat or micropolar model

Classical continuum mechanics considers material continua as simple point-continua with points having three displacement-degrees of freedom, and the response of a material to the displacement of its points is characterized by a symmetric Cauchy stress tensor presupposing that the transmission of loads through surface elements is uniquely determined by a force vector, neglecting couples. Such a model may be insufficient for the description of certain physical phenomena. Non-classical behaviour due to microstructural effects is observed most in regions of high strain gradients, e.g. at notches, holes or cracks.

The Cosserat model is one of the most prominent extended continuum models. It has emerged from the seminal work of the brothers Francois and Eugene Cosserat at the turn of the last century ("Theorie des corps deformables." 1909, Review 1912, english translation by D. Delphenich 2007). They attempted to unify field theories embracing mechanics, optics and electro-dynamics through a common principle of least action (Euclidean action). Their main aim was to produce the correct general form of the energy for the variational problem. Postulating the invariance of energy under Euclidean transformations they were able to derive the equations of balance of forces and balance of angular momentum in a geometrically exact format. However, they never wrote down any constitutive equations.

Compared to classical linear elasticity the model features three additional, independent degrees of freedom, related to the rotation of each particle which need not coincide with the macroscopic rotation of the continuum at the same point. In the simplest isotropic case, one coupling constant, here called Cosserat couple modulus $\mu_c\ge 0$ and three internal length scale parameters need to be determined/measured in addition to the two classical Lame-constants. In the following, I concentrate first on the static linear setting with quadratic free energy. In this linear setting, the model is in fact already given by the german scientist W. Voigt in 1887: "Theoretische Studien über die Elasticitätsverhältnisse der Krystalle. I. Ableitung der Grundgleichungen aus der Annahme mit Polarität begabter Moleküle". Also P. Duhem: "Le potentiel thermodynamique et la pression hydrostatique" (1893) had noticed that various phenomena which seemed incompatible with classical continuum mechanics could be described as effects of direction, and he suggested that materials be visualized as sets of points having vectors attached to them, i.e., oriented or polar media. However, for historical precision, the date of birth of a polar continuum is the year 1686, when Jakob Bernoulli introduced angular momentum as a postulate, independent of balance of momentum. (C.Truesdell: "Die Entwicklung des Drallsatzes", ZAMM 44, 1964)

One of the essential features of polar continua is that the stress tensor is not necessarily symmetric, and the balance of angular momentum equation has to be modified accordingly. All theories in which the stress tensor is not symmetric can be regarded as polar-continua. The non-symmetry of the stress tensor appears also if higher order deformation gradients are included in the free energy, instead of only the first order gradients. Both such theories typically predict a size-effect, meaning that smaller samples of the same material behave relatively stiffer than larger samples. This is an experimental fact, but completely neglected in the classical approach. It implies that some of the additional parameters in the Cosserat model define a length-scale present in the material.

The Cosserat model may be posed in a variational format as a two-field minimization problem for the usual displacement u and the three entries of the infinitesimal microrotation A, which is an element of the Lie-algebra so(3). Find here a concise description of the linear elastic, static Cosserat model.

A good summary of its special features as compared to classical linear elasticity is given on the webpage of Prof. R. Lakes with the precaution to correctly identify the material moduli. Useful is also the short introduction by Prof. S. Forest. Prof. A.C. Eringen has complemented the theory by introducing micro-inertia and renamed it subsequently micropolar theory. In static elasticity, "Cosserat" and "micropolar" may be used interchangeably.

The so-called indeterminate couple stress model (Koiter-Mindlin model) appears formally by setting the Cosserat couple modulus $\mu_c$ to infinity and is, in fact, a special higher order gradient continuum where the higher derivatives act only on the continuum rotation. I have put together some information on the Koiter-

Mindlin model here. In general, the higher the value of \mu_c, the closer is the Cosserat rotation to the continuum (macroscopic) rotation.

As often the case, notation is a nightmare. Unfortunately, in earlier works of Eringen and others, the interpretation of elastic constants have been misleading with the consequence of giving erroneous parameter ranges for positive definiteness. This has been corrected by S.C. Cowin (“An incorrect inequality in micropolar elasticity theory”, ZAMP 1970) and the more recent book by Eringen. Find here the relation of constants and the correct range for positive definiteness of the elasticity tensor for the linear isotropic Cosserat model.

It is impossible to do justice to all those who helped rediscover the Cosserat theory starting in the 1950s. An important milestone was led in the 1968 Freudenstadt-IUTAM-Symposium on the “Mechanics of Generalized Continua” edited by Prof. E. Kröner (Springer book). There, one may find all the masters of the subject up to that time and appreciate the hopes which were tied to this type of theory but which were only partially fulfilled. Notably two points were already stated as the aims of this conference: “The bridging of the gap between microscopic (or atomic) research on mechanics on one hand, and the phenomenological (or continuum mechanical) approach on the other hand.” and (to clarify) “the physical interpretation and the relation to actual material behaviour of the quantities and laws introduced into the new theories, together with applications.” Both topics are not completely solved today.

One can also still recommend to read Truesdell/Noll: Non-Linear Field Theories of Mechanics, Handbuch der Physik, Vol. III/3, Sect. 98: “Polar elastic materials” which summarizes the knowledge up to 1965 together with a more detailed historical account.

The mathematical analysis of linear micropolar models is fairly well established with a wealth of analytical solutions for boundary value problems, existence and uniqueness theorems and continuous dependence results. It is usually based on a uniform positivity assumption on the free energy that sets it apart from linear elasticity in that Korn‘s inequality is traditionally not needed. Prof. D. Iesan (Iassy, Romania) has contributed greatly to this field and the book by Ciarletta/Iesan “Non-classical elastic solids” (Longman 1993) testifies to this endeavour.

Turning to the geometrically exact Cosserat model we deal with exact rotations i.e. the non-commutative Lie-group SO(3). The reader should note that the geometrically exact development can already be found in the original treatment of the Cosserat brothers, only they did not specify any constitutive relations (i.e. stress-strain relations or free energy function W). A brief description in modern notation can be found here, where for simplicity the curvature energy has been taken as a uni-constant approximation. A very readable account of extended continuum models in direct tensor notation has been given by Prof. G. Capriz: “Continua with Microstructure” (Springer 1989).

There is also very interesting work of Prof. J.F. Pommaret connecting the ideas of the Cosserat brothers with Cartan’s torsion (Cartan, according to his own acknowledgement, was inspired by the brothers Cosserat and their new type of continuum) and recent group theoretical methods, see: François Cosserat et le secret de la théorie mathématique de l'élasticité, Annales des Ponts et Chaussées, 82, 1997, 59-66 (in French). Prof. J.F. Pommaret also investigated the life of the brothers Cosserat. He establishes (in the following, phrases in cursive are freely adapted passages from J.F. Pommaret, Lie-Pseudogroups and Mechanics, Gordon and Breach Science, 1988, according to J.F. Pommaret this book is “explaining for the first time the work of the brothers Cosserat within the framework of the formal theory of Lie pseudogroups pioneered by D.C. Spencer”):

Francois Cosserat, born 26. November 1852 in Douai, Ecole Polytechnique in 1870 and Ecole Nationale des Ponts et Chausses in 1870. Third class civil engineer in 1875. Marriage and one daughter in 1878. Railway second class civil engineer (northern zone) in 1879. First class civil engineer in 1883. Chevalier de la legion d'honneur in 1893. Second class chief civil engineer 1895. Officier d'Academie in 1896. Vice-president de la Societe Mathematique de France in 1912. President de la Societe Mathematique de France in 1913. Died March 22, 1914.

A short biography of the younger brother Eugene Cosserat, also by Prof. J.F. Pommaret:

Eugene Cosserat, born 4. March 1866 in Amiens, Ecole Normale Superieure in 1883 and assistant astronomer at the Toulouse observatory in 1886. Docteur des Sciences Mathematiques in 1889 and adjoint astronomer at the Toulouse observatory. Professor of differential and integral calculus at the University of Toulouse in 1895. Laureate of the Institute (Prix Poncelet) in 1899. Head of the Toulouse observatory in 1908. Died May 31, 1931.

Eugene‘s doctor's degree, in the spirit of Gaston Darboux, deals with an extension of Plücker's concepts that is used in order to study the infinitesimal properties of spaces created by circles.

Between 1895 and 1910, the two brothers published together a series of Notes in the Comptes Rendus de l'Academie des Sciences and long accounts or comments in famous textbooks and treatises. It seems that the main "creative" mechanical ideas (the most striking behind the amount of computations) have been furnished by Francois, while Eugene, burdened by management tasks at the Observatory of Toulouse, was just rectifying the computations. Indeed when one gets the basic definitions and provides the variational methods, the main problem is to give them a physical meaning. Also, Eugene was still in a good position for publishing the results after the death of his brother. However, nothing on mechanics was published after 1914. Meanwhile, it is painful to know that, when Francois asked for a position in mechanics at Ecole Polytechnique, in competition with Hadamard and Jouguet, somebody pointed out doubts on his participation in this work in collaboration, and finally Jouguet was chosen.

The latest work done in common by the brothers is the translation and extension of an Historical survey on mechanics in the Encyclopedie des Sciences Mathematiques published in 1915, after an initial version by the German scientist A. Voss.

In fact, among their contemporaries, only Poincare (electron theory), Picard and Cartan appreciated the work done by the brothers. But it is only in Germany that the brothers got disciples like Karl Heun who quoted them with emphasis in the German edition of the Encyclopedie des Sciences Mathematiques and studied their work in a seminar at Karlsruhe in 1909. Then, Relativity Theory and Quantum Physics overtook this period in science and the work of the Cosserats was nearly forgotten except for Sudria: “Contribution a la theorie de l'action euclidienne”. In the 1950s the work of E. and F. Cosserat was beginning to be rediscovered (in its simpler linearized version) partly because of the use of liquid crystals. (end of adaption of Pommaret)

Cosserat theory is fundamentally nonlinear through the appearance of the non-abelian group SO(3). When considering planar problems, however, the axis of rotation is fixed, and we deal with a much simpler abelian SO(2); only one rotation angle survives. In this sense, the planar model is not at all representative of Cosserat behaviour. Even in the linear case, the planar problem annihilates 2 Cosserat curvature constants and may lead to wrong conclusions concerning different curvature expressions.

It is fair to mention that the Cosserat model is still controversially discussed. The introduction to I.A. Kunin‘s book on microstructure is well summarizing the problems. A recent comment on the submission of a paper treating Cosserat media nicely illustrates the state of affairs as perceived by many engineers today:

“Reviewer #1: I cannot recommend this paper for publication in the International Journal of ** for a variety of reasons. First, the micropolar model has been around for several decades but yet there is no compelling evidence for the use of such models. Nor have any of the material moduli which characterize the model been measured. Also, there are problems with regard to the prescription of boundary conditions for the microrotation, etc.”

In my opinion, any Cosserat paper must indeed stand against this criticism! When arguing with this fundamental criticism one should carefully distinguish between the infinitesimal and the finite-strain model. The criticism concerns the infinitesimal isotropic elastic model and it is true, that in this case, the chosen parameters in the literature have mostly no sound basis! This is due to the special coupling between displacements and infinitesimal rotations in the linear model. In many standard textbooks e.g. by A.C. Eringen (Microcontinuum Field Theories I: Foundations and Solids, Vol. 1) and W. Nowatzki (Theory of Asymmetric Elasticity) it is assumed a priori that the free energy of the linear Cosserat solid is uniformly positive definite. In tud-preprint 2409 I have shown that under this assumption, the linear Cosserat model is physically inconsistent in certain cases. A similar remark, however, does not apply to the finite strain Cosserat model since there the coupling can be of a different nature. Nevertheless, by assuming a weaker than traditional curvature energy in the linear model the physical consistency can be re-established, which I have done in joint work with Dr. Jena Jeong (tud-preprint 2550).

It is true that the Cosserat model may degenerate into a mere boundary layer theory in which, moreover, the identified Cosserat parameters would depend on the applied boundary conditions. Whether this happens depends on the used constitutive assumptions. It can be avoided in the linear case by assuming Neumann (free) boundary conditions for the microrotations together with a weak curvature expression and in the finite strain case by assuming \mu_c=0.

The parameters of the Cosserat model must be size-independent in order to make sense at all as physical material parameters. Indeed, otherwise one would get different parameters for the same copper material, say, for small or large samples. The condition on the parameter range determining size-independent material moduli in the linear isotropic Cosserat model can be given based on a posteriori inspection of analytical solutions. It leads to the reduction from 6 to 4 independent parameters: exactly the weakest curvature assumption is retrieved. This is a new observation of the author with possibly important consequences.

Summarizing, it is necessary to distinguish between different fields of applications for Cosserat media, each with different level of success of predictive power and practical usefulness:

The Cosserat approach as a numerical implementational device

Here, the aim is finally to compute classical elasticity solutions. Rotations appear as an intermediary step of the implementation which then need to be condensed out and/or the (numerical) asymmetry of the stresses gets penalized. This idea goes back at least to the work of Reissner (linear elasticity) who wanted to approximate balance of forces and balance of angular momentum on an equal footing, while standard codes approximate balance of forces but satisfy balance of angular momentum on the discrete level exactly (symmetric discrete stresses). It corresponds to replace the Kirchhoff-Love shell by the Reissner-Mindlin shear-deformable shell. The idea was transferred to nonlinear elasticity based on the Biot-stretch tensor (Hughes, Simo, Bufler, Sansour). In this case, it corresponds to a Cosserat model with zero internal length scale. Whether or not penalisation is necessary in the nonlinear, isotropic case has been answered in tud-preprint 2518. This approach is also useful when considering a coupling with dipole-moments. Similarly, the fourth order indeterminate couple stress model can be relaxed to a second-order Cosserat model.

Cosserat media as regularization devices for computations in elasto-plasticity where shear-banding occurs, also in geo-mechanics for the same purpose

This seems to be the most simple setting: the appearing Cosserat parameters can be taken as numerical tuning values, error estimates and convergence analysis can be performed. For certain ranges of parameters, the Cosserat model approximates in a stable manner classical elasticity and elasto-plasticity. Thus the Cosserat approach is only numerically motivated.

Here you find some recent papers treating these aspects: tud-preprint 2412, tud-preprint 2468, tud-preprint 2470, tud-preprint 2520, tud-preprint 25. Notice that the Cosserat model conveys the shear band a definite width instead of an ill-defined shear band width in the non-polar case causing mesh-dependent results in FEM-simulations. This use of the Cosserat model goes back to Prof. H.B. Mühlhaus and Prof. R. de Borst in the early 1990s. Presently it is also investigated in the group of Prof. W. Ehlers, Stuttgart. In the linearized context they use

\mu_c=\mu and a simplified, uni-constant curvature energy (Thesis W. Volk 1999 and B. Scholz 2007) for regularization of softening phenomena.

**Cosserat media as a replacement medium for a granular assembly**
A huge body of work has been dedicated to this problem. The relevant values for material parameters are not really clear since in these applications typically a multifold of physical processes takes place at the same time, like elasticity, plasticity, viscous relaxation etc. But it is experimentally validated that particle rotations are an important factor in the development of shear bands in granular materials. Tentatively, one can say that the Cosserat couple modulus \mu_c would then be related to the rolling stiffness between individual grains. Similar considerations apply to masonry and blocky structures. These topics are further elaborated in the groups of Prof. W. Ehlers and Prof. P. Steinmann (microplane modelling).

**Cosserat media as homogenized versions of structural elements, discrete mass-spring systems and periodic microstructure**
Here it seems to be possible to relate the beam structures geometry to various material moduli. Notably, the smallest distance between grid points can be related to the internal length scale in the Cosserat model. But care has to be exercised throughout. Representative investigations have been carried out by Prof. Z.P. Bazant. I am not using the Cosserat model with this view in mind.

**Cosserat media as a model for the prediction of size-effects in foam like structures (like bones) or cellular materials**

This approach has been championed by Lakes, see above. He determines Cosserat parameters by careful size experiments. His values seem to be the only consistent choice of Cosserat parameters ever given for the linear isotropic model. Interestingly, his values have been rejected by a prominent proponent of the micropolar model because Lakes values make the Cosserat free energy only positive semi-definite instead of some supposed positive definiteness. A careful mathematical inspection of analytical solution reveals, however, that Lakes parameter range is a must in order to avoid certain unphysical stiffening behaviour of the Cosserat model for very small samples. Find a discussion of the physical problem involved and the well-posedness for Lakes value here: tud-preprint 2550. A connection with infinitesimal conformal invariance is discussed in tud-preprint 2558 and tud-preprint 2559. Recently, much has been done in the group of Prof. P. Onck on cellular solids (Thesis C. Tekoglu). There, experiments have been replaced by numerical homogenization schemes. Similar methods are employed in Prof. S. Diebels group.

**Cosserat media as a model for materials with random microstructure and homogenization**
This is a task that is not yet completely finished. A lot of work has been done by Prof. S. Forest on this subject. The author hopes to come up with a yet other answer to this problem in the future.

**Cosserat media as a model to describe defective elastic crystals, independent lattice rotations, plasticity and size-effects**
An elastic crystal in the continuum approximation is a point continuum, but the rotations of particles, or the independent rotation of the atomic lattice cannot be represented in such an approximation. The inclusion of this interaction was the main motivation for the early development of W. Voigt, see above. However, albeit much attention has been devoted to this interpretation, the answer was negative for the linear Cosserat continuum, see e.g., W. Diepholder et al.: "The Cosserat continuum, a model for grain rotations in metals?"

The geometrically exact Cosserat elastoplastic problem has been investigated, among others, in the group of Prof. P. Steinmann and Prof. C. Tsakmakis (TU-Darmstadt, Ph.D-Thesis P. Grammenoudis 2003) as well as by Prof. S. Forest. Generalized mechanical continuum theories and their application in defects and microstructures are investigated by Dr. M. Lazar (TU-Darmstadt).

A dislocated single crystal is another example of a Cosserat continuum for which lattice curvature is due to geometrically necessary dislocations. As a rule in three-dimensions the dislocation density equals torsion. In this direction a novel approach has been taken by me in first considering the geometrically exact Cosserat problem with finite rotations and then setting, contrary to all previous approaches, the Cosserat couple modulus $\mu_c=0$. This gives a completely new physical model. Look at tud-preprint 2518, tud-preprint 2414, tud-preprint 2373 for some of its intriguing aspects. Upon linearization of this Cosserat model with $\mu_c=0$ one obtains classical linear Cauchy elasticity!

**The anisotropic finite strain Cosserat model**
The general anisotropic constitutive law has been derived as early as 1964 by Kessel. However, since determination of constants is crucial for the success of the Cosserat model, and since even the isotropic constants are still being discussed it is of not much use to try to use the most general representation. A small step into a geometrically exact transversely isotropic model has been taken recently with the novelty of interpreting the Cosserat rotation as rotation of a fiber in a fiber/matrix compound which is only possible by choosing $\mu_c=0$.

**Cosserat media as a three-dimensional parent model for the rigorous derivation of shell and plate models**
This is a relatively recent undertaking but it allows to give rigorous derivations for otherwise ad hoc mechanical plate models like the well-known Reissner-Mindlin membrane-bending plate. Here you find some relevant material:

Finite strain Gamma-convergence.

Reissner-Mindlin infinitesimal strain Gamma-convergence. Interestingly, the Cosserat brothers where motivated by shell-theory (triedre cache=orthogonal three-frame of the surface, tangentially attached at each point) to propose their independent orthogonal three-frame (the triedre mobile in their language), which is our Cosserat rotation.

**The direct shell models based on Cosserat kinematics with large rotations**
This is one of the success stories of Cosserat type modelling. Here, the aim is not so much to introduce a new physical shell model but to consistently include large rotations (necessary for shells) in the model. In general one is dealing with a shear-deformable thin shell depending on a number of Cosserat directors. The model can either feature a full Cosserat rotation (inclusion of in-plane drill) or neglect in-plane drill. Such models represent the state of the art for geometrically exact thin shells. A lot of work on this topic has been done in the group of Prof. W. Pietraszkiewicz (Gdansk, Poland) and e.g. by Prof. E. Ramm (U. Stuttgart), Prof. K. Wisniewski (IPPT Warsaw), Prof. M.B. Rubin (Technion, Haifa) and Prof. F. Gruttmann (TU-Darmstadt). In my habilitation thesis I have derived and analysed the geometrically exact shell formulations with large rotations and I gave the first existence theorems for plates. While it is clear in principle how to extend the formulation and analysis to initially curved shells based on an effective shell-gradient formulation this is subject of current research with M. Birsan.

I have said nothing on the important dynamic problems of Cosserat elasticity. The differences between classical linear elasticity and experiment appear particularly in dynamic problems involving elastic vibrations of high frequencies and short wave-length. The reason for these differences lies in the microstructure of the material which exerts a special influence. The linear dynamic Cosserat model predicts dispersion relations (the wave speed depends on the wave frequency: impossible in classical linear elasticity, but an experimental fact) and predicts new rotational waves. These special waves, however, have not been observed yet to my knowledge. Wave propagation phenomena with experimentally observed dispersion relations are treated e.g. by Prof. Vladimir I. Erofeyev.:“Wave processes in Solids with Microstructures.” The analysis, however, is practically restricted to linear models. Note that the geometrically exact Cosserat model with \mu_c=0 does not seem to predict new wave forms.

Another interesting field is micropolar fluid flow with an eye towards modelling turbulence and augmenting the Navier-Stokes equations. The constitutive equations for a linear polar fluid where first introduced by H. Grad in 1952. The equations are used e.g. to simulate blood flow. An exposition is given by Prof. Lukaszewicz: “Micropolar Fluids: Theory and Applications.” New mathematical problems have been tackled by Prof. M. Ruzicka.