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A Leonard-Sanders-Budiansky-Koiter-Type Nonlinear Shell Theory with a Hierarchy of Transverse-Shearing Deformations

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Summary

A detailed exposition on a refined nonlinear shell theory that is suitable for nonlinear limit-point buckling analyses of practical laminated-composite aerospace structures is presented. This shell theory includes the classical nonlinear shell theory attributed to Leonard, Sanders, Koiter, and Budiansky as an explicit proper subset that is obtained directly by neglecting all quantities associated with higher-order effects such as transverse-shearing deformation. This approach is used in order to leverage the exisiting experience base and to make the theory attractive to industry. In addition, the formalism of general tensors is avoided in order to expose the details needed to fully understand and use the theory in a process leading ultimately to vehicle certification.

The shell theory presented is constructed around a set of strain-displacement relations that are based on "small" strains and "moderate" rotations. No shell-thinness approximations involving the ratio of the maximum thickness to the minimum radius of curvature are used and, as a result, the strain-displacement relations are exact within the presumptions of "small" strains and "moderate" rotations. To faciliate physical insight, these strain-displacement relations are presented in terms of the linear reference-surface strains, rotations, and changes in curvature and twist that appear in the classical "best" first-approximation linear shell theory attributed to Sanders, Koiter, and Budiansky. The effects of transverse-shearing deformations are included in the strain-displacement relations and kinematic equations by using analyst-defined functions to describe the through-the-thickness distributions of transverse-shearing strains. This approach yields a wide range of flexibility to the analyst when confronted with new structural configurations and the need to analyze both global and local response phenomena, and it enables a building-block approach to analysis. The theory also uses the three-dimensional elasticity form of the internal virtual work to obtain the symmetrical effective stress resulants that appear in classical nonlinear shell theory attributed to Leonard, Sanders, Koiter, and Budiansky. The principle of virtual work, including "live" pressure effects, and the surface divergence theorem are used to obtain the nonlinear equilibrium equations and boundary conditions.

A key element of the shell theory presented herein is the treatment of the constitutive equations, which include thermal effects. The constitutive equations for laminated-composite shells are derived without using any shell-thinness approximations, and simplified forms and special cases are discussed that include the use of layerwise zigzag kinematics. In addition, the effects of shell-thinness approximations on the constitutive equations are presented. It is noteworthy that none of the shell-thinness approximations appear outside of the constitutive equations, which are inherently approximate. Lastly, the effects of "small" initial geometric imperfections are introduced in a relatively simple manner, and a resume′ of the fundamental equations are given in an appendix. Overall, a hierarchy of shell theories that are amenable to the

prediction of global and local responses and to the development of generic design technology are obtained in a detailed and unified manner.

Major Symbols

The primary symbols used in the present study are given as follows.

$\hat{a}_1, \hat{a}_2$	unit-magnitude base vector fields of the shell reference surface shown in figure 1
A	area of shell reference surface (see figure 1), in^2
A_1, A_2	metric coefficients of the shell reference surface (see equation (1)
$A^k_{11}, A^k_{12}, A^k_{16}, A^k_{22}, A^k_{26}, A^k_{66}$	shell stiffnesses defined by equation (101)
$A_{11}, A_{12}, A_{16}, A_{22}, A_{26}, A_{66}$	shell membrane stiffnesses (see equation (B31)), lb/in.
A_{44}, A_{45}, A_{55}	shell transverse-shearing stiffnesses (see equation (B32)), lb/in.
$B_{11}, B_{12}, B_{16}, B_{22}, B_{26}, B_{66}$	shell coupling stiffnesses (see equation (B31)), lb
c_1, c_2, c_3	tracers used to indentify various shell theories (see equation (53))
$\overline{C}_{ij}$	transformed shear stiffnesses appearing in equation (85b), psi
$[\mathbf{C}_{ij}]$	constitutive matrices defined by equations (87)
ds	differential arc length define by equation (1), in.
$D_{11}, D_{12}, D_{16}, D_{22}, D_{26}, D_{66}$	shell bending and twisting stiffnesses (see equation (B31)), in.-lb
$[\mathbf{d}_0], [\mathbf{d}_1], [\mathbf{d}_2]$	matrices defined by equations (26), (27), and (55)
$[\mathbf{d}^i_0], [\mathbf{d}^i_1], [\mathbf{d}^i_2]$	matrices defined by equations (116) - (117)
$e^o_{11}, e^o_{22}, e^o_{12}$	linear deformation parameters defined by equations (8)

$F_1(\xi_3)$, $F_2(\xi_3)$	analyst-defined functions that specifiy the through-the-thickness distributions of the transverse-shearing strains (see equations (3)), in.
$\mathcal{F}_{11}, \mathcal{F}_{12}, \mathcal{F}_{21}, \mathcal{F}_{22}, \{\mathcal{F}_1\}, \{\mathcal{F}_2\}$	work-conjugate stress resultants defined by equations (20c) and (20d), lb
$G(\xi_3)$	function defining the through-the-thickness temperature variation (see equation (90))
$g_{11}^{jk}, g_{12}^{jk}, g_{22}^{jk}$	shell thermal coefficients defined by equation (109)
h	shell-wall thickness, in.
$\mathcal{h}/\mathcal{R}$	maximum shell thickness divided by the minimum principal radius of curvature
$h_{11}^{k}, h_{12}^{k}, h_{22}^{k}$	shell thermal coefficients defined by equation (108)
k_{44}, k_{45}, k_{55}	transverse-shear correction factors appearing in equation (B32)
$[\mathbf{k}_0], [\mathbf{k}_1], [\mathbf{k}_2], [\mathbf{k}_{11}], [\mathbf{k}_{12}], [\mathbf{k}_{22}]$	matrices defined by equations (28) and (29)
$M_{11}, M_{12}, M_{21}, M_{22}$	bending stress resultants defined by equations (13), in-lb/in.
$\mathcal{M}_{11}, \mathcal{M}_{12}, \mathcal{M}_{22}, \{\mathcal{M}\}$	work-conjugate bending stress resultants defined by equation (20b), in-lb/in.
$M_1(\xi_2), M_{12}(\xi_2)$	applied loads on edge ξ_1 = constant (see figure 2), in-lb/in.
$M_{21}(\xi_1), M_2(\xi_1)$	applied loads on edge ξ_2 = constant (see figure 2), in-lb/in.
$\hat{n}$	unit-magnitude base vector field perpendicular to the shell reference surface, as depicted in figure 1
$\hat{N}$	unit-magnitude vector field perpendicular to the shell reference-surface boundary curve and $\hat{n}$, as depicted in figure 1
$N_{11}, N_{12}, N_{21}, N_{22}$	membrane stress resultants defined by equations (13), lb/in.
$\mathcal{N}_{11}, \mathcal{N}_{12}, \mathcal{N}_{22}, \{\mathcal{N}\}$	work-conjugate membrane stress resultants defined by equation (20a), lb/in.
$N_1(\xi_2)$	applied load on edge ξ_1 = constant (see figure 2), lb/in.
$N_2(\xi_1)$	applied load on edge ξ_2 = constant (see figure 2), lb/in.

$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$	effective tractions defined by equations (59), psi
$\mathcal{P}_1^i, \mathcal{P}_2^i, \mathcal{P}_3^i$	effective tractions defined by equations (122), psi
$q_1, q_2, q_3, \{\mathbf{q}\}$	applied surface tractions (see equations (32)), psi
q_1^D, q_2^D, q_3^D	dead-load part of applied surface tractions (see equations (32)), psi
q_3^L	live-load part of applied surface tractions (see equations (32)), psi
q_1^i, q_2^i, q_3^i	tractions associated with interactions between live pressure and initial geometric imprefections, defined by equation (122d), psi
$Q_1(\xi_2)$	applied load on edge ξ_1 = constant (see figure 2), lb/in.
$Q_2(\xi_1)$	applied load on edge ξ_2 = constant (see figure 2), lb/in.
Q_{13}, Q_{23}	transverse-shear stress resultants defined by equations (13), lb/in.
$\mathcal{Q}_{13}, \mathcal{Q}_{23}, \{\mathcal{Q}\}$	work-conjugate transverse-shear stress resultants defined by equation (20e), lb/in.
$\tilde{Q}_{13}, \tilde{Q}_{23}$	stress resultants defined by equations (59), lb/in.
$\overline{Q}_{ij}$	transformed, reduced (plane stress) stiffnesses of classical laminated-shell and laminated-plate theories (see equation (85a)), psi
$Q_{11}^{ijk}, Q_{12}^{ijk}, Q_{16}^{ijk}, Q_{22}^{ijk}, Q_{26}^{ijk}, Q_{66}^{ijk}$	shell stiffnesses defined by equation (103)
R_1, R_2	principal radii of curvature of the shell reference surface along the ξ_1 and ξ_2 coordinate directions, respectively, in.
$R_{11}^{jk}, R_{12}^{jk}, R_{16}^{jk}, R_{22}^{jk}, R_{26}^{jk}, R_{66}^{jk}$	shell stiffnesses defined by equation (102)
$S_1(\xi_2)$	applied load on edge ξ_1 = constant (see figure 2), lb/in.
$S_2(\xi_1)$	applied load on edge ξ_2 = constant (see figure 2), lb/in.
$[\mathbf{S}_0], [\mathbf{S}_1], [\mathbf{S}_2], [\mathbf{S}_3], [\mathbf{S}_4], [\mathbf{S}_5]$	matrices defined by equations (17)

u_1, u_2, u_3	displacements of material points comprizing the shell reference surface (see equations (3)), in.
U_1, U_2, U_3	displacements of shell material points (see equations (3)), in.
$w^i(\xi_1, \xi_2)$	known, measured or assumed, distribution of reference-surface initial geometric imperfections measured along a vector normal to the reference surface at a given point, in.
$X_{44}^{ijk}, X_{45}^{ijk}, X_{55}^{ijk}$	shell stiffnesses defined by equation (104)
$Y_{44}^{ijk}, Y_{45}^{ijk}, Y_{55}^{ijk}$	shell stiffnesses defined by equation (105)
z_1, z_2	quantities defined as $1+\frac{\xi_3}{R_1}$ and $1+\frac{\xi_3}{R_2}$, respectively, and used in equations (89) and (99)
Z	quantity defined as $z_1 + z_2 + \frac{1}{2}(z_2 - z_1)^2$ and used in equations (89) and (99)
$Z_{44}^{ijk}, Z_{45}^{ijk}, Z_{55}^{ijk}$	shell stiffnesses defined by equation (106)
$\bar{\alpha}_{ij}$	transformed coefficients of thermal expansion appeing in equation (85a), $^{o}F^{-1}$
$\gamma_{13}^{o}, \gamma_{23}^{o}, \{\boldsymbol{\gamma}^{o}\}$	transverse-shearing strains evaluated at the shell reference surface (see equation (16c))
Γ_{12}	transverse shear function defined by equation (5g)
$\delta e_{11}^{o}, \delta e_{22}^{o}, \delta e_{12}^{o}$	variations of the linear deformation parameters defined by equations (23)
$\delta\varepsilon_{11}, \delta\varepsilon_{22}, \delta\gamma_{12}, \delta\gamma_{13}, \delta\gamma_{23}, \delta\varepsilon_{33}$	virtual strains appearing in equation (14b)
$\delta\varepsilon_{11}^{o}, \delta\varepsilon_{22}^{o}, \delta\gamma_{12}^{o}, \{\boldsymbol{\delta\varepsilon}^{o}\}$	virtual membrane strains defined by equation (22a) and (53)
$\delta\gamma_{13}^{o}, \delta\gamma_{23}^{o}, \{\boldsymbol{\delta\gamma}^{o}\}$	virtual transverse-shearing strains appearing in equation (22c)
$\delta\varphi_1, \delta\varphi_2, \delta\varphi$	virtual rotations of the shell reference surface about the ξ_1-, ξ_2-, and ξ_3-axes, respectively, defined by equations (23), radians

$\delta\chi^{o}_{11}$, $\delta\chi^{o}_{22}$, $\delta\chi^{o}_{12}$, $\{\boldsymbol{\delta\chi}^{o}\}$	vector of virtual bending strains defined by equation (22b), in^{-1}
$\delta\mathcal{W}_{E}$	external virtual work per unit area of shell reference surface defined by equation (32a), lb/in.
$\delta\mathcal{W}^{B}_{E}$	external virtual work per unit length of the applied tractions acting on the boundary curve ∂A that encloses the region A (see figure 1 and equation (33a)), lb
$\delta\mathcal{W}^{B}_{E1}$, $\delta\mathcal{W}^{B}_{E2}$	external virtual work per unit length of shell reference surface boundary defined by equations (33), lb
$\delta\mathcal{W}_{I}$	internal virtual work per unit area of shell reference surface defined by equations (19), lb/in.
$\delta\tilde{\mathcal{W}}_{I}$	internal virtual work per unit volume of shell defined by equations (14), psi
$\delta\mathcal{W}^{B}_{I}$	internal virtual work per unit length of shell reference surface boundary defined by equation (47), lb
$\delta\mathcal{W}^{B}_{I1}$, $\delta\mathcal{W}^{B}_{I2}$	internal virtual work per unit length of shell reference surface boundary defined by equations (48), lb
δu_1, δu_2, δu_3, $\{\boldsymbol{\delta u}\}$	virtual displacements of the shell reference surface about the ξ_1-, ξ_2-, and ξ_3-axes, respectively, in. (see equation (27a))
ε_{11}, ε_{22}, γ_{12}, γ_{13}, γ_{23}, ε_{33}	shell strains defined by equations (5)
ε^{o}_{11}, ε^{o}_{22}, γ^{o}_{12}, $\{\boldsymbol{\varepsilon}^{o}\}$	reference-surface normal and shearing strains defined by equations (6) and (51)
κ^{o}_{11}, κ^{o}_{22}, $2\kappa^{o}_{12}$	changes in reference surface curvature and torsion defined by equations (7), in^{-1}
ρ_{11}, ρ_{22}	radii of geodesic curvature of the shell reference surface coordinate curves ξ_1 and ξ_2, respectively, in.

$\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{33}$	shell stresses, psi
τ	parameter used to identify second-order terms in equations (100)
ξ_1, ξ_2, ξ_3	curvilinear coordinates of the shell, as depicted in figure 1
$\varphi_1, \varphi_2, \varphi$	linear rotation parameters for the shell reference surface defined by equations (4), radians
$\overset{o}{\chi}_{11}, \overset{o}{\chi}_{22}, 2\overset{o}{\chi}_{12}, \{\overset{o}{\boldsymbol{\chi}}\}$	linear deformation parameters defined by equations (8) and (16b), in.$^{-1}$
$\Theta, \hat{\Theta}$	function describeing the pointwise change in temperature from a uniform reference state (see equations (90)), °F
$\{\boldsymbol{\Theta}_k\}, \{\bar{\boldsymbol{\Theta}}_k\}$	thermal quantities defined by equations (88) and (92), respectively
∂A	curve bounding area of shell reference surface (see figure 1), in.

Introduction

Classical plate and shell theories have played an important role in the design of high-performance aerospace structures for many years. As a result, familiarity with these theories is generally widespread throughout the aerospace industry, and a great amount of resources has gone into validating their use in design. Perhaps the best-known classical shell theory is the one attributed to A. E. H. Love[1, 2] that was re-derived by Reissner[3] in 1941. This shell theory has become known worldwide as the Love-Kirchhoff classical thin-shell theory. This particular shell theory, as presented by Reissner, has some deficiencies that were later addressed by Sanders, Budiansky, and Koiter[4-6] to obtain what is generally deemed as the "best" first-approximation classical thin-shell theory. This shell theory was later extended to include the effects of geometric nonlinearities by Leonard,[7] Sanders,[8] Koiter,[9,10] and Budiansky.[11] For the most part, these theories are focused on shells made of isotropic materials.

As the need for improved structural performance has increased, new materials and design concepts have emerged that require refined plate and shell theories in order to predict adequately the structural behavior. For example, a sandwich plate with fiber-reinforced face plates and a relatively flexible core, either of which may have embedded electromechanical actuation layers, is a structure that typically requires a refined theory. Similarly, efforts made over the last 20 to 30 years to reduce structural weight or to enable active shape control have resulted in thin-walled, relatively flexible designs that require nonlinear theories to predict accurately responses such as buckling and flutter.

Many refined plate and shell theories have been developed over the past 50 or so years that are classified as equivalent single-layer, layer-wise, zigzag, and variational asymptotic theories. Detailed historical accounts of these theories are beyond the scope of the present study and can be found in references 12-80. Each of these theories has its own merits and range of validity associated with a given class of problems, and the choice of which theory to use depends generally on the nature of the response characteristics of interest. For the most part, these theories have not yet found wide acceptance in standard industry design practices because of the extensive experience base with classical theories, the relatively limited amount of validation studies, and the increased complexity that designers usually try to avoid. In general, validation studies associated with stuctures made from exotic state-of-the-art materials are very expensive if experiments are involved. Moreover, there are usually many more structural parameters that must be examined in order to understand the design space, compared to the number of parameters that characterize the behaviors of the more commonplace metallic structures.

The present study is concerned with the development of refined shell theories that include the classical shell theories as well-defined, explicit proper subsets. Herein, the term "explicit proper subset" means that the equations of a particular classical shell theory appear directly when all quantities associated with higher-order effects, such as transverse shearing deformations, are neglected. In contrast, the terms "implicit subset" and "contained implicitly" are used to indicate cases where the equations of a particular classical shell theory can be recovered by using a transformation of the fundamental unknown response functions. This interest in refined shell theories that include the classical shell theories as well-defined, explicit subsets is motivated by the need for design-technology and certification technology development that takes full advantage of the existing experience base. For example, legacy codes used by industry that have undergone extensive, expensive experimental validation over many years can be enhanced to address issues associated with new materials and design concepts with a high degree of confidence. Moreover, this approach appears to avoid undesirable computational ill-conditioning effects.[61] Likewise, experience and insight gained in the development and use of nondimensional parameters[81-165] to characterize the very broad response spectrum of laminated-composite plates and shells can be retained and extended with the high degree of confidence needed to design and certify aerospace vehicles. Furthermore, the development and use of nondimensional parameters have a high potential to impact the development of scaling technologies that can be used to design sub-scale experiments for validation of new analysis methods and for flight certification of aerospace vehicles (e. g., see reference 147).

Of the many refined theories for plates and shells discussed in references 12-80, several are particularly relevant to the present study.[166-212] In an early 1958 paper by Ambartsumian,[166] a general equivalent single-layer, linear theory of anisotropic shells was derived that presumes parabolic through-the-thickness distributions for the transverse-shearing stresses. Subsequent integration of the corresponding strain-displacement relations is shown to yield expressions for the displacement fields that include those of the classical theory of shells explicitly as a proper subset. Six equilibrium equations are also used that involve the asymmetrical shearing and twisting stress resultants that are obtained by integrating the shearing stresses across the shell thickness. A similar derivation was presented later by Ambartsumian[167] for shallow shells in 1960.

Likewise, Tomashevski[168] used a similar approach to derive the corresponding equations for buckling of orthotropic cylinders in 1966.

In 1969, Cappelli et.al.[169] presented equations for orthotropic shells of revolution that are based on Sanders'[8] linear shell theory and that include the effects of transverse-shear deformations. In those equations, the two rotations of a material line element that is perpendicular to the shell reference surface are used as fundamental unknowns and, as a result, the corresponding equations of classical shell theory do not appear explicitly as a proper subset. In contrast, Bhimaraddi[170] presented linear equations for vibration analysis of isotropic circular cylindrical shells in 1984 that include parabolic through-the-thickness distributions for the transverse-shearing stresses and contain Flügge's equations[171] as an explicit proper subset. It is noteworthy to recall that Flügge's equations retains terms of second order in the ratio of the maximum thickness to the minimum radius of curvature that is used in the shell-thinness approximations.

Also in 1984, Reddy[172] presented a linear first-order transverse-shear-deformation theory for doubly curved, laminated-composite shells that extends Sanders' original work[4] by including the two rotations of a material line element that is perpendicular to the shell reference surface that are used as fundamental unknowns and by introducing constitutive equations that relate the transverse-shear stress resultant to the transverse shearing strains. Similarly, in 1985, Reddy and Liu[173] extended Reddy's previous shear-deformation theory for doubly curved, laminated-composite shells by including parabolic through-the-thickness distributions for the transverse-shearing stresses. Like Cappelli et.al.,[169] the equations in references 172 and 173 do not contain the equations of Sanders' shell theory as an explicit proper subset.

Soldatos[174-178] presented a refined shear-deformation theory for isotropic and laminated-composite non-circular cylindrical shells during 1986-1992. This particular theory includes parabolic through-the-thickness distributions for the transverse-shearing stresses and contains the equations of Love-Kirchoff classical shell theory as an explicit proper subset. Additionally, only five independent unknown functions are present in the kinematic equations, like first-order transverse-shear deformation theories. In 1989, Bhimaraddi et al.[179] presented a derivation for a shear-deformable shell finite element that is based on kinematics that include parabolic through-the-thickness distributions for the transverse-shearing strains in addition to the kinematics based on the hypothesis originally used by Love.[1, 2] Likewise, in 1992, Touratier[180] presented a generalization of the theories discussed herein so far that combines parabolic through-the-thickness distributions for the transverse-shearing strains and the classical Love-Kirchhoff linear shell theory. Specifically, following his earlier work on plates (see reference 181), Touratier appended the Love-Kirchhoff kinematics for shells undergoing axisymmetric deformations with a transverse-shear deformation term that uses a somewhat arbitrary function of the through-the-thickness coordinate to define the distributions of the transverse-shearing stresses. The arbitrariness of this function is limited by the requirement that the corresponding transverse shearing stresses satisfy the traction boundary conditions on the bounding surfaces of the shell. This process yields general functional representations for the transverse shearing stresses, much like that of Ambartsumian,[166, 167, 182] that are specified by the analyst a priori. Moreover, by specifying the appropriate shear-deformation functions, the first-order and refined theories that have uniform and parabolic shear-stress distributions, respectively, discussed herein previously are obtained as special cases. Touratier[181, 183] also presented results based on using sinusoidal through-the-thickness distributions for the transverse-shearing strains that are similar to those used earlier by Stein and Jegley.[184, 185] A similar formulation for shallow shells was given by Sklepus[186] in 1996, which includes thermal effects. Additionally, in 1992, Soldatos[187] presented a refined shear-deformation theory for circular cylindrical shells that is similar to general formulation of Touratier[180] but utilizes only four unknown functions in the kinematic equations and also accounts for transverse normal strains. Later, in 1999, Lam et.al.[188] determined the vibration modes of thick laminated-composite cylindrical shells by using a refined theory that includes parabolic through-the-thickness distributions for the transverse-shearing strains in addition to the kinematics of classical Love-Kirchhoff shell theory. In 2001, Fares & Youssif[189] derived an improved first-order shear-deformation nonlinear shell theory, with the Sanders-type kinematics used by Reddy,[172] that uses a mixed variational principle to obtain stresses that are continuous across the shell thickness. A similar theory was also derived by Zenkour and Fares[190] in 2001 for laminated cylindrical shells.

Recently, Mantari et.al.[191] presented a linear theory for doubly curved shallow shells, made of laminated-composite materials, that is similar in form to the derivation given by Reddy and Liu,[173] but uses the form of the kinematics used by Touratier[180] in 1992 and discussed previously herein. In contrast to Touratier's work, the theory given by Mantari et.al. contains the linear equations of the Donnell-Mushtari-Vlasov[192] shell theory as an explicit proper subset. In addition, Mantari et al. use a special form of the functions used to specify the through-the-thickness distributions of the transverse-shearing strains that contains a "tuning" parameter. This parameter is selected to maximize the transverse flexibility of a given laminate construction. A similar derivation, but with an emphasis on a different form of the functions used to specify the through-the-thickness distributions of the transverse-shearing strains was presented by Mantari et al.[193, 194] in 2012. Very recently, Viola et.al.[195] presented a general high-order, linear, equivalent single-layer shear-deformation theory for shells that contains many of the theories described herein previously as special cases.

Several shell theories have been derived over the past 25 to 30 years that utilize layerwise kinematics to enhance an equivalent single-layer theory without introducing additional unknown independent functions that lead to boundary-value problems of higher order. In 1991, Librescu & Schmidt[196] presented a general theory of shells that appends the kinematics of first-order shear-deformation shell theory with layerwise functions that are selected to yield displacement and stress continuity at layer interfaces. However, the traction boundary conditions at the top and bottom shell surfaces are not satisfied. In addition, the theory includes the effects of relatively small-magnitude geometric nonlinearities. In contrast, to a standard first-order shear-deformation shell theory, this theory of Librescu & Schmidt has a twelveth-order system of equations governing the response.

Later, in 1993 and 1995, Soldatos and Timarci[197, 198] presented, and applied, a general formulation for cylindrical laminated-composite shells, similar to that of Touratier,[180] that includes five independent unknown functions in the kinematic equations and a discussion about incorporating layerwise zigzag kinematics into the functions used to specify the through-the-thickness distribution of transverse shearing stresses. Similarly, in 1993 and 1995, Jing and Tzeng[199, 200] presented, and applied, a refined theory for laminated-composite shells that is based on the kinematics of first-order transverse-shear-deformation shell theory appended with zigzag layer-displacement functions, and on assumed independent transverse-shearing stress fields that satisfy traction continuity at the layer interfaces. A mixed variational approach is used to obtain the compatibility equations for transverse-shearing deformations in addition to the equilibrium equations and boundary conditions. Although, the theory captures the layerwise deformations and stresses, it has only seven unknown functions in the kinematic equations, regardless of the number of layers. In addition, the theory includes the exact form of the shell curvature terms appearing in the strain-displacement relations of elasticity theory and in the usual, general definitions of the stress resultants for shells. Likewise, in 1993, a general theory for doubly curved laminated-composite shallow shells was presented by Beakou and Touratier[201] that incorporates zigzag layer-displacement functions and that has only five unknown functions in the kinematic equations, regardless of the number of layers. Similar work was presented by Ossadzow, Muller, Touratier, and Faye[202, 203] in 1995. Moreover, Shaw and Gosling[204] extended the theory of Beakou and Touratier[201] in 2011 to include non-shallow, deep shells.

In 1994, He[205] presented a general linear theory of laminated-composite shells that has the three unknown reference-surface displacement fields of classical Love-Kirchhoff shell theory and two additional ones that are selected to satisfy continuity of displacements and stresses at layer interfaces and the traction boundary conditions at the two bounding surfaces of a shell. Similarly, Shu[206] presented a linear theory for laminated-composite shallow shells in 1996 that also satisfies continuity of displacements and stresses at layer interfaces and the traction boundary conditions at the two bounding surfaces of a shell. Shu's theory, however, contains the equations of the Donnell-Mushtari-Vlasov[192] shell theory as an explicit proper subset. In 1997, Shu[207] extended this theory to include nonshallow shells, with the classical Love-Kirchhoff shell theory as an explicit proper subset.

Cho, Kim, and Kim[208] presented a refined theory for laminated-composite shells in 1996 that is based on the kinematics of first-order transverse-shear-deformation shell theory appended with zigzag layer-displacement functions. In contrast to the theory of Jing and Tzeng,[199] this theory uses a displacement formulation to enforce traction continuity at the layer interfaces. In 1999, Soldatos & Shu[209] presented a stress analysis method for doubly curved laminated shells that is based on their earlier work (e.g., see references 197, 198, 206, and 208), which includes five unknown independent functions in the kinematic equations and two general functions that are used to specify the through-the-thickness distribution of transverse shearing stresses. The two functions that are used to specify the through-the-thickness distribution of transverse shearing stresses are determined by applying the two equilibrium equations of elasticity theory that relate the transverse-shearing stresses to the stresses acting in the tangent plane to obtain a system of ordinary differential equations for the two unknown functions. This approach yields solutions for the two functions in each shell layer. The constants of integration are determined by enforcing continuity of displacements and stresses at layer interfaces and the traction boundary conditions at the two bounding surfaces of a shell.

The present study is also concerned primarily with the development of refined nonlinear shell theories. Several previous works relevant to the present study are given by references 210-220 and 72. Specifically, in 1987, Librescu[210] presented a general theory for geometrically perfect, elastic, anisotropic, multilayer shells of general shape, using the formalism of general tensors, that utilizes a mixed variational approach to obtain the equations governing the shell response. These equations include continuity conditions for stresses and displacements at the layer interfaces. In his theory, the displacement fields are expanded in power series with respect to the through-the-thickness shell coordinate, and then substituted into the three-dimensional, nonlinear Green-Lagrange strains of elasticity theory. This step yields nonlinear strain-displacement relations with no restrictions placed on the size of the displacement gradients, and it contains the classical "small finite deflection" theory of Koiter[9] and the classical "small strain-moderate rotation" theory of Sanders[8] as special cases. A wide range of refined geometrically nonlinear shell theories can be obtained from Librescu's general formulation, each of which is based on the number of terms retained in the power series expansions. In 1988, Librescu and Schmidt[211] presented a similar derivation for a general theory of shells, again using the formalism of general tensors, that uses Hamilton's variational principle to derive the equations of motion and boundary conditions. This particular derivation did not yield continuity conditions for stresses and displacements at the layer interfaces, and focused on geometric nonlinearities associated with "small" strains and "moderate" rotations. Likewise, Schmidt and Reddy[212] presented a general first-order shear-deformation theory for elastic, geometrically perfect anisotropic shells in 1988, following an approach similar to Librescu and Schmidt, for "small" strains and "moderate" rotations that includes uniform through-the-thickness normal strain. Their derivation also uses the formalism of general tensors. Another similar presentation and an assessment of the theory was given by Palmerio, Reddy, and Schmidt in 1990.[213, 214] In contrast to the theory of Librescu and Schmidt, the theory of Schmidt and Reddy utilizes a simpler set of strain-displacement relations which neglects nonlinear rotations about the vector field normal to the reference surface. Moreover, the dynamic version of the principle of virtual displacements is used to obtain the corresponding equations of motion and boundary conditions. Furthermore, the classical "small" strain and "moderate" rotation theories given by Leonard,[7] Sanders,[8] and Koiter[9,10] are contained in the Schmidt-Reddy theory implicitly; that is, they can be obtain by using a change of independent variables in the equations governing the response.

In 1991, Carrera[215] presented a first-order shear-deformation theory for buckling and vibration of doubly curved laminated-composite shells that includes the Flügge-Lur´e-Byrne equations[216] as an explicit proper subset. This particular set of classical equations for doubly curved shells also retains terms of second order in the ratio of the maximum thickness to the minimum radius of curvature that is used in the shell-thinness approximations. The geometric nonlinearities used by Carrera are identical to those of the Donnell-Mushtari-Vlasov shell theory, as presented by Sanders.[8] In 1992 and 1993, Simitses and Anastasiadis[217, 218] presented a refined nonlinear theory for moderately thick, laminated-composite, circular cylindrical shells that includes geometric nonlinearity associated with "small" strains and "moderate" rotations, like that given by Sanders,[8] and initial geometric imperfections. The theory is based on cubic through-the-thickness axial and circumferential displacements and constant through-the-thickness normal displacements, and neglects rotations about the vector field normal to the reference surface. Moreover, the classical theory of Sanders[8] is contained as an implicit subset. Also in 1992, Soldatos[219] presented a refined nonlinear theory for geometrically perfect laminated-composite cylindrical shells with a general non-circular cross-sectional profile. Soldatos' theory presumes parabolic through-the-thickness distributions for the transverse-shearing stresses and contains the equations of Love-Kirchoff classical shell theory as an explicit proper subset. Additionally, only five unknown independent functions are present in the kinematic equations, like first-order transverse-shear deformation theories. The geometric nonlinearity correspond to the "small" strains and "moderate" rotations of Sanders,[8] with rotations about the normal vector field neglected. The equations of motion and boundary conditions are obtained by using Hamilton's variational principle.

Several years later, in 2008, Takano[220] presented a nonlinear theory for geometrically perfect, anisotropic, circular cylindrical shells. His theory uses the full geometric nonlinearity possessed by the three-dimensional Green-Lagrange strains of elasticity theory, and the equilibrium equations and boundary conditions are obtained by applying the principle of virtual work. Moreover, the theory is formulated as a first-order shear deformation theory and, when linearized, includes Flügge's equations[171] as an explicit proper subset. Takano's work is similar to that presented in 1991 by Carrera[215] for doubly curved shells, but includes a higher degree of nonlinearity in the strain-displacement relations. In 2009, Pirrera and Weaver[72] presented a nonlinear, first-order shear-deformation theory for geometrically perfect anisotropic shells that uses the rotations of material line elements normal to the reference surface as fundamental unknowns. In their theory, the full geometric nonlinearity possessed by the three-dimensional Green-Lagrange strains of elasticity theory is used and expressed in terms of linear strains and rotations, and their products. Additionally, the equations of motion used in their theory are based on momentum balance of a differential shell element, as opposed to being determined from a variational principle. Moreover, the equations of motion are linear, which appears to be inconsistent with a geometrically nonlinear theory.

The literature review given previously herein reveals a need for a detailed exposition on a refined nonlinear shell theory that is suitable for nonlinear limit-point buckling analyses of practical aerospace structures made of laminated-composites that utilize advanced structural design concepts. A major goal of the present study is to supply this exposition. Another goal is to focus on a shell theory that includes the classical nonlinear shell theories as explicit proper subsets in order to leverage the exisiting experience base and to make the theory attractive to industry. To accomplish these goals, the formalism of general tensors is avoided in order to expose the details needed to fully understand and use the theory in a process leading ultimately to vehicle certification. In addition, the analysis is simplified greatly by focusing on the many practical cases that can be addressed by using principal-curvature coordinates. The key to accomplishing these goals is the form of the strain-displacement relations.

The strain-displacement relations used in the present study are a subset of those derived in reference 221, that are useful for nonlinear limit-point buckling analyses. These strain-displacement relations are based on "small" strains and "moderate" rotations, and are presented first, along with a description of the shell geometry and kinematics. Moreover, the strain-displacement relations are presented in terms of the linear reference-surface strains, rotations, and changes in curvature and twist that appear in the classical "best" first-approximation linear shell theory attributed to Sanders[4], Koiter[5], and Budiansky.[6] The effects of transverse-shearing deformations are included in the strain-displacement relations and kinematic equations by using the approach of Touratier[180] in which the through-the-thickness distributions of transverse-shearing strains are represented by analyst-defined functions. Additionally, no shell-thinness approximations involving the ratio of the maximum thickness to the minimum radius of curvature are used and, as a result, the strain-displacement relations are exact within the presumptions of "small" strains and "moderate" rotations. This approach yields a wide range of flexibility to the analyst when confronted with new structural configurations and the need to analyze both global and local response phenomena. Next, the usual asymmetrical shell stress resultants that are obtained by integrating the stresses across the shell thickness are defined, and the three-dimensional elasticity form of the internal virtual work is given and used to obtain the corresponding symmetrical effective stress resulants that appear in classical nonlinear shell theory attributed to Leonard,[7] Sanders,[8] Koiter,[9,10] and Budiansky.[11] Afterward, the principle of virtual work, including "live" pressure effects, and the surface divergence theorem are used to obtain the nonlinear equilibrium equations and boundary conditions. Then, the thermoelastic constitutive equations for laminated-composite shells are derived without using any shell-thinness approximations. Simplified forms and special cases of the constitutive equations are also discussed that include the use of layerwise zigzag kinematics. In addition, the effects of shell-thinness approximations on the constitutive equations are presented. It is noteworthy to mention that none of the shell-thinness approximations discussed in the present study appear outside of the constitutive equations, which are inherently approximate due to the fact that their specification requires experimentally determined quantities that are often not known precisely. Lastly, the effects of "small" initial geometric imperfections are introduced in a relatively simple manner, and a resume′ of the fundamental equations are given in an appendix. Overall, a hierarchy of shell theories are obtained in a detailed and unified manner that are amenable to the prediction of global and local responses and to the development of generic design technology.

Geometry and Coordinate Systems

The equations governing the nonlinear deformations of doubly curved shells are presented subsequently in terms of the orthogonal, principal-curvature, curvilinear coordinates (ξ_1,ξ_2,ξ_3) that are depicted in figure 1 for a generic shell reference surface A. Associated with each point **p** of the reference surface, with coordinates $(\xi_1,\xi_2,0)$, are three perpendicular, unit-magnitude vector fields $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$. The vectors $\hat{a}_1$ and $\hat{a}_2$ are tangent to the ξ_1- and ξ_2-coordinate curves, respectively, and reside in the tangent plane at the point **p**. The vector $\hat{n}$ is tangent to the ξ_3-coordinate curve at point **p** and perpendicular to the tangent plane. The metric coefficients of the reference surface, also known as coefficients of the first fundamental form, are denoted by the functions $A_1(\xi_1,\xi_2)$ and $A_2(\xi_1,\xi_2)$ that appear in the equation

$$(ds)^2 = (A_1 d\xi_1)^2 + (A_2 d\xi_2)^2 \tag{1}$$

where ds is the differential arc length between two infinitesimally neighboring points of the surface, **p** and **q**. This class of parametric coordinates permits substantial simplification of the shell equations and has many practical applications.

Principal-curvature coordinates form an orthogonal coordinate mesh and are identified by examining how the vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$ change as the coordinate curves are traversed by an infinitesimal amount. In particular, at every point **q** that is infinitesimally close to point **p** there is another set of vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$ with similar attributes; that is, the vectors $\hat{a}_1$ and $\hat{a}_2$ are orthogonal and tangent to the ξ_1- and ξ_2-coordinate curves at **q**, respectively, and reside in the tangent plane at the point **q**. Likewise, vector $\hat{n}$ is tangent to the ξ_3-coordinate curve at point **q** and perpendicular to the tangent plane at point **q**. Next, consider the finite portion of the tangent plane at point **p** shown in figure 1. Because of the identical properties of the vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$ at every point of the surface, an identical, corresponding planar region exists at point **q**. Therefore, the vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$ at point **q** can be obtained by moving the vectors $\hat{a}_1$, $\hat{a}_2$, and $\hat{n}$ at point **p** to point **q**. In addition, the plane region at point **p** moves into coincidence with the corresponding plane region at point **q** as the surface is traversed from point **p** to point **q**. During this process, the plane region at point **p** undergoes roll, pitch, and yaw (rotation about the normal line to the surface) motions. The roll and pitch motions are caused by surface twist (torsion) and curvature, respectively. The yaw motion is associated with the geodesic curvature of the surface curve traversed in going from point **p** to **q**. When a principal-curvature coordinate curve is traversed in going from point **p** to **q**, the planar region at point **p** undergoes only pitch and yaw motions as it moves into coincidence with the corresponding region at point **q**. Rolling motion associated with local surface torsion does not occur. This attribute simplifies greatly the mathematics involved in deriving a shell theory.

In the shell-theory equations presented herein, the functions $R_1(\xi_1,\xi_2)$ and $R_2(\xi_1,\xi_2)$ denote the principal radii of curvature of the shell reference surface along the ξ_1 and ξ_2 coordinate directions, respectively. Similarly, the functions $\rho_{11}(\xi_1,\xi_2)$ and $\rho_{22}(\xi_1,\xi_2)$ denote the radii of geodesic curvature of the shell reference surface coordinate curves ξ_1 and ξ_2, respectively. Discussions of these quantities are found in the books by Weatherburn[222], Eisenhart,[223] Struik,[224] and Kreyszig.[225] These functions are related to the metric coefficients by the equations

$$\frac{1}{\rho_{11}} = -\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \xi_2} \tag{2a}$$

$$\frac{1}{\rho_{22}} = \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \xi_1} \tag{2b}$$

Kinematics and Strain-Displacement Relations

The kinematics and strain-displacement relations presented in this section are special cases of those given in reference 221. These equations were derived on the presumption of "small" strains and "moderate" rotations. Moreover, no shell-thinness approximations were used in their derivation. In the subsequent presentations, the term "tangential" refers to quantities associated with the tangent plane at a given point of the shell reference surface. In contrast, the term "normal" refers to quantities perpendicular to the tangent plane at that given point.

The tangential and normal displacement fields of a material point (ξ_1, ξ_2, ξ_3) of a shell are expressed in orthogonal principal-curvature coordinates as

$$U_1(\xi_1, \xi_2, \xi_3) = u_1 + \xi_3[\varphi_1 - \varphi\varphi_2] + F_1(\xi_3)\overset{\circ}{\gamma}_{13} \tag{3a}$$

$$U_2(\xi_1, \xi_2, \xi_3) = u_2 + \xi_3[\varphi_2 + \varphi\varphi_1] + F_2(\xi_3)\overset{\circ}{\gamma}_{23} \tag{3b}$$

$$U_3(\xi_1, \xi_2, \xi_3) = u_3 - \tfrac{1}{2}\xi_3(\varphi_1^2 + \varphi_2^2) \tag{3c}$$

where $U_1, U_2,$ and U_3 are the displacement-field components in the ξ_1-, ξ_2-, and ξ_3-coordinate directions, respectively. The functions $u_1(\xi_1, \xi_2)$ and $u_2(\xi_1, \xi_2)$ are the corresponding tangential displacements of the reference-surface material point $(\xi_1, \xi_2, 0)$, and $u_3(\xi_1, \xi_2)$ is the normal displacement of the material point $(\xi_1, \xi_2, 0)$. In addition, the functions $\varphi_1(\xi_1, \xi_2)$, $\varphi_2(\xi_1, \xi_2)$, and $\varphi(\xi_1, \xi_2)$ are linear rotation parameters that are given in terms of the reference-surface displacements by

$$\varphi_1(\xi_1, \xi_2) = \frac{u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial \xi_1} \tag{4a}$$

$$\varphi_2(\xi_1, \xi_2) = \frac{u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial \xi_2} \tag{4b}$$

$$\varphi(\xi_1, \xi_2) = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}\right) \tag{4c}$$

The functions $F_1(\xi_3)\overset{\circ}{\gamma}_{13}(\xi_1, \xi_2)$ and $F_2(\xi_3)\overset{\circ}{\gamma}_{23}(\xi_1, \xi_2)$ define the transverse-shearing strains. In particular, $F_1(\xi_3)$ and $F_2(\xi_3)$ are analyst-defined functions that specifiy the through-the-thickness distributions of the transverse-shearing strains, and are selected to satisfy the traction-free boundary conditions on the transverse-shear stresses at the bounding surfaces of the shell given by the coordinates $\left(\xi_1, \xi_2, \pm\frac{h}{2}\right)$. In addition, $F_1(\xi_3)$ and $F_2(\xi_3)$ are selected to satisfy the conditions $U_1(\xi_1, \xi_2, 0) = u_1(\xi_1, \xi_2)$ and $U_2(\xi_1, \xi_2, 0) = u_2(\xi_1, \xi_2)$. Thus, from equations (3) it follows that $F_1(0) = F_2(0) = 0$.

The nonlinear strain-displacement relations obtained from reference 221 are given as follows. The normal strains are

$$\varepsilon_{11}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[\varepsilon^{\circ}_{11}\left(1 + \frac{\xi_3}{R_1}\right) + \xi_3\kappa^{\circ}_{11} + F_1(\xi_3)\frac{1}{A_1}\frac{\partial\gamma^{\circ}_{13}}{\partial\xi_1} - F_2(\xi_3)\frac{\gamma^{\circ}_{23}}{\rho_{11}}\right] \tag{5a}$$

$$\varepsilon_{22}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[\varepsilon^{\circ}_{22}\left(1 + \frac{\xi_3}{R_2}\right) + \xi_3\kappa^{\circ}_{22} + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{22}} + F_2(\xi_3)\frac{1}{A_2}\frac{\partial\gamma^{\circ}_{23}}{\partial\xi_2}\right] \tag{5b}$$

$$\varepsilon_{33}(\xi_1, \xi_2, \xi_3) = 0 \tag{5c}$$

and the shearing strains are

$$\gamma_{12}(\xi_1, \xi_2, \xi_3) = \frac{\frac{1}{2}\gamma^{\circ}_{12}\left[\left(1 + \frac{\xi_3}{R_1}\right)^2 + \left(1 + \frac{\xi_3}{R_2}\right)^2\right] + \xi_3\kappa^{\circ}_{12}\left[2 + \frac{\xi_3}{R_1} + \frac{\xi_3}{R_2}\right] + 2\Gamma_{12}(\xi_1, \xi_2, \xi_3)}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)} \tag{5d}$$

$$\gamma_{13}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[F_1'(\xi_3)\left(1 + \frac{\xi_3}{R_1}\right) - \frac{F_1(\xi_3)}{R_1}\right]\gamma^{\circ}_{13} \tag{5e}$$

$$\gamma_{23}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[F_2'(\xi_3)\left(1 + \frac{\xi_3}{R_2}\right) - \frac{F_2(\xi_3)}{R_2}\right]\gamma^{\circ}_{23} \tag{5f}$$

where

$$2\Gamma_{12} = \left(1 + \frac{\xi_3}{R_1}\right)\left[F_1(\xi_3)\frac{1}{A_2}\frac{\partial\gamma^{\circ}_{13}}{\partial\xi_2} - F_2(\xi_3)\frac{\gamma^{\circ}_{23}}{\rho_{22}}\right] + \left(1 + \frac{\xi_3}{R_2}\right)\left[F_2(\xi_3)\frac{1}{A_1}\frac{\partial\gamma^{\circ}_{23}}{\partial\xi_1} + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{11}}\right] \tag{5g}$$

From equations (5e) and (5f), it follows that $\gamma_{13}(\xi_1, \xi_2, 0) = F_1'(0)\gamma^{\circ}_{13}$ and $\gamma_{23}(\xi_1, \xi_2, 0) = F_2'(0)\gamma^{\circ}_{23}$. Thus, it is convenient to scale $F_1(\xi_3)$ and $F_2(\xi_3)$ to give $F_1'(0) = F_2'(0) = 1$. For this scaling, γ°_{13} and γ°_{23} are the transverse-shearing strains at the shell reference surface.

In equations (5), ε°_{11}, ε°_{22}, and γ°_{12} are the reference-surface normal and shearing strains, which are given in terms of the linear strain and rotation parameters $e^{\circ}_{11}, e^{\circ}_{22}, e^{\circ}_{12}, \varphi_1, \varphi_2$, and φ by

$$\varepsilon^{\circ}_{11}(\xi_1, \xi_2) = e^{\circ}_{11} + \frac{1}{2}\left[(e^{\circ}_{11})^2 + (e^{\circ}_{12} + \varphi)^2 + \varphi_1^2\right] \qquad (6a)$$

$$\varepsilon^{\circ}_{22}(\xi_1, \xi_2) = e^{\circ}_{22} + \frac{1}{2}\left[(e^{\circ}_{12} - \varphi)^2 + (e^{\circ}_{22})^2 + \varphi_2^2\right] \qquad (6b)$$

$$\gamma^{\circ}_{12}(\xi_1, \xi_2) = 2e^{\circ}_{12} + e^{\circ}_{11}(e^{\circ}_{12} - \varphi) + e^{\circ}_{22}(e^{\circ}_{12} + \varphi) + \varphi_1\varphi_2 \qquad (6c)$$

Likewise, the changes in reference surface curvature and torsion κ°_{11}, κ°_{22}, and $2\kappa^{\circ}_{12}$ caused by deformation are given by

$$\kappa^{\circ}_{11}(\xi_1, \xi_2) = \chi^{\circ}_{11} - \frac{e^{\circ}_{11}}{R_1} \qquad (7a)$$

$$\kappa^{\circ}_{22}(\xi_1, \xi_2) = \chi^{\circ}_{22} - \frac{e^{\circ}_{22}}{R_2} \qquad (7c)$$

$$2\kappa^{\circ}_{12}(\xi_1, \xi_2) = 2\chi^{\circ}_{12} - e^{\circ}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad (7c)$$

where χ°_{11}, χ°_{22}, and χ°_{12} are linear strain parameters associated with bending and twisting of the shell reference surface. The linear strain parameters are given in terms of the reference-surface displacements and linear rotations by

$$e^{\circ}_{11}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{u_3}{R_1} \qquad (8a)$$

$$e^{\circ}_{22}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{u_3}{R_2} \tag{8b}$$

$$2e^{\circ}_{12}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \tag{8c}$$

$$\chi^{\circ}_{11}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} \tag{8d}$$

$$\chi^{\circ}_{22}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} \tag{8e}$$

$$2\chi^{\circ}_{12}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2}\frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}} - \varphi\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{8f}$$

In these equations, $e^{\circ}_{11}, e^{\circ}_{22},$ and $2e^{\circ}_{12}$ are recognized as the linear reference-surface strains given by Sanders in reference 4. Likewise, $\chi^{\circ}_{11}, \chi^{\circ}_{22},$ and $2\chi^{\circ}_{12}$ are the linear bending-strain measures given by Sanders.

To arrive at the particular form of the nonlinear strain-displacement relations used in the present study, equations (5a), (5b), and (5d) are first re-arranged to get

$$\varepsilon_{11}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[\varepsilon^{\circ}_{11} + \xi_3\left(\kappa^{\circ}_{11} + \frac{\varepsilon^{\circ}_{11}}{R_1}\right) + F_1(\xi_3)\frac{1}{A_1}\frac{\partial \gamma^{\circ}_{13}}{\partial \xi_1} - F_2(\xi_3)\frac{\gamma^{\circ}_{23}}{\rho_{11}}\right] \tag{9a}$$

$$\varepsilon_{22}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[\varepsilon^{\circ}_{22} + \xi_3\left(\kappa^{\circ}_{22} + \frac{\varepsilon^{\circ}_{22}}{R_2}\right) + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{22}} + F_2(\xi_3)\frac{1}{A_2}\frac{\partial \gamma^{\circ}_{23}}{\partial \xi_2}\right] \tag{9b}$$

$$\gamma_{12}(\xi_1, \xi_2, \xi_3) = \frac{\frac{1}{2}\gamma^{\circ}_{12}\left[\left(1 + \frac{\xi_3}{R_1}\right) + \left(1 + \frac{\xi_3}{R_2}\right) + \frac{1}{2}\left(\frac{\xi_3}{R_2} - \frac{\xi_3}{R_1}\right)^2\right]}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)}$$

$$+ \frac{\xi_3\left(\kappa^{\circ}_{12} + \frac{1}{4}\gamma^{\circ}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right)\left[\left(1 + \frac{\xi_3}{R_1}\right) + \left(1 + \frac{\xi_3}{R_2}\right)\right] + 2\Gamma_{12}(\xi_1, \xi_2, \xi_3)}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)} \tag{9c}$$

where the identity

$$\left(\frac{\xi_3}{R_2} - \frac{\xi_3}{R_1}\right)^2 = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left[\xi_3\left(1 + \frac{\xi_3}{R_2}\right) - \xi_3\left(1 + \frac{\xi_3}{R_1}\right)\right] \tag{10}$$

has been used to obtain this particular form of these equations. As pointed out by Koiter,[6, 9-10] terms involving a reference-surface strain divided by a principal radius of curvature are extremely small and can be added or neglected without significantly altering the fidelity of the strain-displacement relations. Thus, it follows that

$$\left\{\begin{matrix} \kappa^{\circ}_{11} + \dfrac{\varepsilon^{\circ}_{11}}{R_1} \\ \kappa^{\circ}_{22} + \dfrac{\varepsilon^{\circ}_{22}}{R_2} \\ 2\kappa_{12} + \dfrac{1}{2}\gamma^{\circ}_{12}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) \end{matrix}\right\} \approx \left\{\begin{matrix} \chi^{\circ}_{11} \\ \chi^{\circ}_{22} \\ 2\chi^{\circ}_{12} \end{matrix}\right\} \equiv \{\chi^{\circ}\} \tag{11}$$

and equations (9) reduce to

$$\varepsilon_{11} = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[\varepsilon^{\circ}_{11} + \xi_3\chi^{\circ}_{11} + F_1(\xi_3)\frac{1}{A_1}\frac{\partial\gamma^{\circ}_{13}}{\partial\xi_1} - F_2(\xi_3)\frac{\gamma^{\circ}_{23}}{\rho_{11}}\right] \tag{12a}$$

$$\varepsilon_{22} = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[\varepsilon^{\circ}_{22} + \xi_3\chi^{\circ}_{22} + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{22}} + F_2(\xi_3)\frac{1}{A_2}\frac{\partial\gamma^{\circ}_{23}}{\partial\xi_2}\right] \tag{12b}$$

$$\gamma_{12} = \frac{\frac{1}{2}\gamma^{o}_{12}\left[\left(1+\frac{\xi_3}{R_1}\right)+\left(1+\frac{\xi_3}{R_2}\right)+\frac{1}{2}\left(\frac{\xi_3}{R_2}-\frac{\xi_3}{R_1}\right)^2\right]+\xi_3\chi^{o}_{12}\left[\left(1+\frac{\xi_3}{R_1}\right)+\left(1+\frac{\xi_3}{R_2}\right)\right]+2\Gamma_{12}}{\left(1+\frac{\xi_3}{R_1}\right)\left(1+\frac{\xi_3}{R_2}\right)} \qquad (12c)$$

Equations (12) and equations (5e) and (5f) constitute the nonzero nonlinear strain displacement relations of the present study. The membrane reference-surface strains defined by equations (6) are identical to those used by Budiansky[11] and Koiter,[9, 10] and contain those used by Sanders[8] as a special case. Likewise, as mentioned before, the linear bending strain measures defined by equations (8d)-(8f) are identical to those used in the "best" first-approximation linear shell theory of Sanders, Budiansky, and Koiter. In contrast to the linear bending strain measures used in classical Love-Kirchhoff shell theory, those given by equations (8d)-(8f) vanish for rigid-body displacements (see reference 4).

Stress Resultants and Virtual Work

In the classical theories of shells, two-dimensional stress-resultant functions are used to represent the actual force per unit length produced by the internal stresses acting on the normal sections, or faces, of a shell given by constant values of the reference-surface coordinates ξ_1 and ξ_2. On an edge given by ξ_1 = constant, the stress resultants are defined as

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ M_{11} \\ M_{12} \\ Q_{13} \end{Bmatrix} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{\xi_3}{R_2}\right) \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \xi_3\sigma_{11} \\ \xi_3\sigma_{12} \\ \sigma_{13} \end{Bmatrix} d\xi_3 \qquad (13a)$$

In these definitions, the middle surface of the shell is used as the reference surface, for convenience. Likewise, on an edge given by ξ_2 = constant, the stress resultants are defined as

$$\begin{Bmatrix} N_{21} \\ N_{22} \\ M_{21} \\ M_{22} \\ Q_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(1+\frac{\xi_3}{R_1}\right) \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \xi_3\sigma_{12} \\ \xi_3\sigma_{22} \\ \sigma_{23} \end{Bmatrix} d\xi_3 \qquad (13b)$$

where σ_{11}, σ_{22}, σ_{12}, σ_{13}, and σ_{23} are stresses and where, in general, $h = h(\xi_1, \xi_2)$ is the shell thickness at the point (ξ_1, ξ_2) of the shell reference surface. Equations (13) show that the stress resultants are not symmetric; that is, $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$, even though the stresses $\sigma_{12} = \sigma_{21}$. As a result of this asymmetry, constitutive equations that are based on equations (13) are typically more complicated that the corresponding equations for flat plates.

To obtain symmetric stress resultant definitions that yield a simple form for the constitutive equations, the internal virtual work is used. The internal virtual work of a three-dimensional solid is given, in matrix form, in terms of the curvilinear coordinates used herein by

$$\iint_A \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \delta\tilde{\mathcal{W}}_I \left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right) d\xi_3 \; A_1 A_2 d\xi_1 d\xi_2 \tag{14a}$$

with

$$\delta\tilde{\mathcal{W}}_I = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}^{\mathbf{T}} \begin{Bmatrix} \delta\varepsilon_{11} \\ \delta\varepsilon_{22} \\ \delta\gamma_{12} \end{Bmatrix} + \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}^{\mathbf{T}} \begin{Bmatrix} \delta\gamma_{13} \\ \delta\gamma_{23} \\ \delta\varepsilon_{33} \end{Bmatrix} \tag{14b}$$

where the superscript T in equation (14b) denotes matrix transposition and where A denotes the reference-surface area of the shell. The functions $\delta\varepsilon_{11}$, $\delta\varepsilon_{22}$, $\delta\gamma_{12}$, $\delta\gamma_{13}$, $\delta\gamma_{23}$, and $\delta\varepsilon_{33}$ in equation (14b) are the virtual strains that are obtained by taking the first variation of the corresponding shell strains. To obtain the form needed for the present shell theory, it is convenient to express the shell strains given by equations (12), (4e), and (4f) in matrix form as

$$\begin{aligned} \left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right) \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} &= [\mathbf{S_0}]\{\boldsymbol{\varepsilon}^{\mathbf{o}}\} + [\mathbf{S_1}]\{\boldsymbol{\chi}^{\mathbf{o}}\} \\ &+ [\mathbf{S_2}]\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\{\boldsymbol{\gamma}^{\mathbf{o}}\} + [\mathbf{S_3}]\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\{\boldsymbol{\gamma}^{\mathbf{o}}\} + [\mathbf{S_4}]\{\boldsymbol{\gamma}^{\mathbf{o}}\} \end{aligned} \tag{15a}$$

$$\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right) \begin{Bmatrix} \gamma_{13} \\ \gamma_{23} \end{Bmatrix} = [\mathbf{S_5}]\{\boldsymbol{\gamma}^{\mathbf{o}}\} \tag{15b}$$

where

$$\{\boldsymbol{\varepsilon}^{o}\} \equiv \begin{Bmatrix} \varepsilon_{11}^{o} \\ \varepsilon_{22}^{o} \\ \gamma_{12}^{o} \end{Bmatrix} = \begin{Bmatrix} e_{11}^{o} + \frac{1}{2}\left[\left(e_{11}^{o}\right)^{2} + \left(e_{12}^{o} + \varphi\right)^{2} + \varphi_{1}^{2}\right] \\ e_{22}^{o} + \frac{1}{2}\left[\left(e_{12}^{o} - \varphi\right)^{2} + \left(e_{22}^{o}\right)^{2} + \varphi_{2}^{2}\right] \\ 2e_{12}^{o} + e_{11}^{o}\left(e_{12}^{o} - \varphi\right) + e_{22}^{o}\left(e_{12}^{o} + \varphi\right) + \varphi_{1}\varphi_{2} \end{Bmatrix} \tag{16a}$$

$$\{\boldsymbol{\chi}^{o}\} \equiv \begin{Bmatrix} \chi_{11}^{o} \\ \chi_{22}^{o} \\ 2\chi_{12}^{o} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_{1}}\frac{\partial\varphi_{1}}{\partial\xi_{1}} - \frac{\varphi_{2}}{\rho_{11}} \\ \frac{1}{A_{2}}\frac{\partial\varphi_{2}}{\partial\xi_{2}} + \frac{\varphi_{1}}{\rho_{22}} \\ \frac{1}{A_{2}}\frac{\partial\varphi_{1}}{\partial\xi_{2}} + \frac{1}{A_{1}}\frac{\partial\varphi_{2}}{\partial\xi_{1}} + \left(\frac{\varphi_{1}}{\rho_{11}} - \frac{\varphi_{2}}{\rho_{22}}\right) + \left(\frac{1}{R_{2}} - \frac{1}{R_{1}}\right)\varphi \end{Bmatrix} \tag{16b}$$

$$\{\boldsymbol{\gamma}^{o}\} \equiv \begin{Bmatrix} \gamma_{13}^{o} \\ \gamma_{23}^{o} \end{Bmatrix} \tag{16c}$$

and where

$$[\mathbf{S_0}] = \begin{bmatrix} \left(1 + \frac{\xi_3}{R_2}\right) & 0 & 0 \\ 0 & \left(1 + \frac{\xi_3}{R_1}\right) & 0 \\ 0 & 0 & \frac{1}{2}\left[\left(1 + \frac{\xi_3}{R_1}\right) + \left(1 + \frac{\xi_3}{R_2}\right) + \frac{1}{2}\left(\frac{\xi_3}{R_2} - \frac{\xi_3}{R_1}\right)^{2}\right] \end{bmatrix} \tag{17a}$$

$$[\mathbf{S_1}] = \xi_3 \begin{bmatrix} 1 + \frac{\xi_3}{R_2} & 0 & 0 \\ 0 & 1 + \frac{\xi_3}{R_1} & 0 \\ 0 & 0 & \frac{1}{2}\left(1 + \frac{\xi_3}{R_1}\right) + \frac{1}{2}\left(1 + \frac{\xi_3}{R_2}\right) \end{bmatrix} \tag{17b}$$

$$\left[\mathbf{S_2}\right] = \left(1 + \frac{\xi_3}{R_2}\right)\begin{bmatrix} F_1(\xi_3) & 0 \\ 0 & 0 \\ 0 & F_2(\xi_3) \end{bmatrix} \tag{17c}$$

$$\left[\mathbf{S_3}\right] = \left(1 + \frac{\xi_3}{R_1}\right)\begin{bmatrix} 0 & 0 \\ 0 & F_2(\xi_3) \\ F_1(\xi_3) & 0 \end{bmatrix} \tag{17d}$$

$$\left[\mathbf{S_4}\right] = \begin{bmatrix} 0 & -\dfrac{F_2(\xi_3)}{\rho_{11}}\left(1 + \dfrac{\xi_3}{R_2}\right) \\ \dfrac{F_1(\xi_3)}{\rho_{22}}\left(1 + \dfrac{\xi_3}{R_1}\right) & 0 \\ \dfrac{F_1(\xi_3)}{\rho_{11}}\left(1 + \dfrac{\xi_3}{R_2}\right) & -\dfrac{F_2(\xi_3)}{\rho_{22}}\left(1 + \dfrac{\xi_3}{R_1}\right) \end{bmatrix} \tag{17e}$$

$$\left[\mathbf{S_5}\right] = \begin{bmatrix} \left(1 + \dfrac{\xi_3}{R_2}\right)\left[F_1'(\xi_3)\left(1 + \dfrac{\xi_3}{R_1}\right) - \dfrac{F_1(\xi_3)}{R_1}\right] & 0 \\ 0 & \left(1 + \dfrac{\xi_3}{R_1}\right)\left[F_2'(\xi_3)\left(1 + \dfrac{\xi_3}{R_2}\right) - \dfrac{F_2(\xi_3)}{R_2}\right] \end{bmatrix} \tag{17f}$$

Taking the first variation of equations (15) gives

$$\begin{aligned} \left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)\begin{Bmatrix} \delta\varepsilon_{11} \\ \delta\varepsilon_{22} \\ \delta\gamma_{12} \end{Bmatrix} &= \left[\mathbf{S_0}\right]\left\{\boldsymbol{\delta\varepsilon}^{\mathbf{o}}\right\} + \left[\mathbf{S_1}\right]\left\{\boldsymbol{\delta\chi}^{\mathbf{o}}\right\} \\ &+ \left[\mathbf{S_2}\right]\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left\{\boldsymbol{\delta\gamma}^{\mathbf{o}}\right\} + \left[\mathbf{S_3}\right]\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left\{\boldsymbol{\delta\gamma}^{\mathbf{o}}\right\} + \left[\mathbf{S_4}\right]\left\{\boldsymbol{\delta\gamma}^{\mathbf{o}}\right\} \end{aligned} \tag{18a}$$

$$\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)\begin{Bmatrix} \delta\gamma_{13} \\ \delta\gamma_{23} \end{Bmatrix} = \left[\mathbf{S_5}\right]\left\{\boldsymbol{\delta\gamma}^{\mathbf{o}}\right\} \tag{18b}$$

Next, substituting equations (18) into equations (14) and performing the through-the-thickness integration yields the internal virtual work as

$$\iint_A \delta\mathcal{W}_I \, A_1 A_2 d\xi_1 d\xi_2 \tag{19a}$$

with

$$\delta\mathcal{W}_I = \{\mathcal{N}\}^T\{\delta\boldsymbol{\varepsilon}^o\} + \{\mathcal{M}\}^T\{\delta\boldsymbol{\chi}^o\} + \{\mathcal{F}_1\}^T\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\{\delta\boldsymbol{\gamma}^o\} + \{\mathcal{F}_2\}^T\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\{\delta\boldsymbol{\gamma}^o\} + \{\mathcal{Q}\}^T\{\delta\boldsymbol{\gamma}^o\} \tag{19b}$$

where

$$\{\mathcal{N}\} = \begin{Bmatrix}\mathcal{N}_{11}\\ \mathcal{N}_{22}\\ \mathcal{N}_{12}\end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_0}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} d\xi_3 \tag{20a}$$

$$\{\mathcal{M}\} = \begin{Bmatrix}\mathcal{M}_{11}\\ \mathcal{M}_{22}\\ \mathcal{M}_{12}\end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_1}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} d\xi_3 \tag{20b}$$

$$\{\mathcal{F}_1\} = \begin{Bmatrix}\mathcal{F}_{11}\\ \mathcal{F}_{12}\end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_2}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} d\xi_3 \tag{20c}$$

$$\{\mathcal{F}_2\} = \begin{Bmatrix}\mathcal{F}_{21}\\ \mathcal{F}_{22}\end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_3}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} d\xi_3 \tag{20d}$$

$$\{\mathcal{Q}\} = \begin{Bmatrix}\mathcal{Q}_{13}\\ \mathcal{Q}_{23}\end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_4}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{Bmatrix} d\xi_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} [\mathbf{S_5}]^{\mathbf{T}} \begin{Bmatrix}\sigma_{13}\\ \sigma_{23}\end{Bmatrix} d\xi_3 \tag{20e}$$

are defined as work-conjugate stress resultants. The relationship of these quantities with those defined by equations (13) are

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ 2\mathcal{N}_{12} \\ \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ 2\mathcal{M}_{12} \end{Bmatrix} = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} + N_{21} + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(M_{12} - M_{21}\right) \\ M_{11} \\ M_{22} \\ M_{12} + M_{21} \end{Bmatrix} \tag{21}$$

which are identical to the effective stress results first defined by Sanders.[4] The virtual reference-surface strains are given by

$$\left\{\boldsymbol{\delta\varepsilon}^{\mathbf{o}}\right\} = \begin{Bmatrix} \delta\varepsilon^{o}_{11} \\ \delta\varepsilon^{o}_{22} \\ \delta\gamma^{o}_{12} \end{Bmatrix} = \begin{Bmatrix} \left(1 + e^{o}_{11}\right)\delta e^{o}_{11} + \left(e^{o}_{12} + \varphi\right)\delta e^{o}_{12} + \varphi_1\delta\varphi_1 + \left(e^{o}_{12} + \varphi\right)\delta\varphi \\ \left(1 + e^{o}_{22}\right)\delta e^{o}_{22} + \left(e^{o}_{12} - \varphi\right)\delta e^{o}_{12} + \varphi_2\delta\varphi_2 - \left(e^{o}_{12} - \varphi\right)\delta\varphi \\ \left(e^{o}_{12} - \varphi\right)\delta e^{o}_{11} + \left(e^{o}_{12} + \varphi\right)\delta e^{o}_{22} + \left(2 + e^{o}_{11} + e^{o}_{22}\right)\delta e^{o}_{12} \\ \quad + \varphi_2\delta\varphi_1 + \varphi_1\delta\varphi_2 + \left(e^{o}_{22} - e^{o}_{11}\right)\delta\varphi \end{Bmatrix} \tag{22a}$$

$$\left\{\boldsymbol{\delta\chi}^{\mathbf{o}}\right\} = \begin{Bmatrix} \delta\chi^{o}_{11} \\ \delta\chi^{o}_{22} \\ \delta2\chi^{o}_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{A_1}\frac{\partial\delta\varphi_1}{\partial\xi_1} - \frac{\delta\varphi_2}{\rho_{11}} \\ \frac{1}{A_2}\frac{\partial\delta\varphi_2}{\partial\xi_2} + \frac{\delta\varphi_1}{\rho_{22}} \\ \frac{1}{A_2}\frac{\partial\delta\varphi_1}{\partial\xi_2} + \frac{1}{A_1}\frac{\partial\delta\varphi_2}{\partial\xi_1} + \left(\frac{\delta\varphi_1}{\rho_{11}} - \frac{\delta\varphi_2}{\rho_{22}}\right) + \delta\varphi\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \end{Bmatrix} \tag{22b}$$

$$\left\{\boldsymbol{\delta\gamma}^{\mathbf{o}}\right\} = \begin{Bmatrix} \delta\gamma^{o}_{13} \\ \delta\gamma^{o}_{23} \end{Bmatrix} \tag{22c}$$

where the variations of the linear deformation parameters are given by

$$\delta e^{o}_{11}\left(\xi_1, \xi_2\right) = \frac{1}{A_1}\frac{\partial\delta u_1}{\partial\xi_1} - \frac{\delta u_2}{\rho_{11}} + \frac{\delta u_3}{R_1} \tag{23a}$$

$$\delta e^{o}_{22}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial \delta u_2}{\partial \xi_2} + \frac{\delta u_1}{\rho_{22}} + \frac{\delta u_3}{R_2} \tag{23b}$$

$$2\delta e^{o}_{12}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} + \frac{\delta u_1}{\rho_{11}} - \frac{\delta u_2}{\rho_{22}} \tag{23c}$$

$$\delta \varphi_1(\xi_1, \xi_2) = \frac{\delta u_1}{R_1} - \frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1} \tag{23d}$$

$$\delta \varphi_2(\xi_1, \xi_2) = \frac{\delta u_2}{R_2} - \frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2} \tag{23e}$$

$$\delta \varphi(\xi_1, \xi_2) = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} + \frac{\delta u_1}{\rho_{11}} + \frac{\delta u_2}{\rho_{22}}\right) \tag{23f}$$

From these expressions, it follows that

$$\begin{aligned}\delta \varepsilon^{o}_{11} = \left[\frac{\varphi_1}{R_1} + \frac{e^{o}_{12} + \varphi}{\rho_{11}}\right]\delta u_1 - \frac{1 + e^{o}_{11}}{\rho_{11}}\,\delta u_2 + \frac{1 + e^{o}_{11}}{R_1}\,\delta u_3 + \left(1 + e^{o}_{11}\right)\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \xi_1} \\ + \left(e^{o}_{12} + \varphi\right)\frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} - \varphi_1\frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1}\end{aligned} \tag{24a}$$

$$\begin{aligned}\delta \varepsilon^{o}_{22} = \frac{1 + e^{o}_{22}}{\rho_{22}}\,\delta u_1 + \left[\frac{\varphi_2}{R_2} - \frac{e^{o}_{12} - \varphi}{\rho_{22}}\right]\delta u_2 + \frac{1 + e^{o}_{22}}{R_2}\,\delta u_3 + \left(e^{o}_{12} - \varphi\right)\frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} \\ + \left(1 + e^{o}_{22}\right)\frac{1}{A_2}\frac{\partial \delta u_2}{\partial \xi_2} - \varphi_2\frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2}\end{aligned} \tag{24b}$$

$$\begin{aligned}\delta \gamma^{o}_{12} = \left[\frac{\varphi_2}{R_1} + \frac{1 + e^{o}_{22}}{\rho_{11}} + \frac{e^{o}_{12} + \varphi}{\rho_{22}}\right]\delta u_1 + \left[\frac{\varphi_1}{R_2} - \frac{e^{o}_{12} - \varphi}{\rho_{11}} - \frac{1 + e^{o}_{11}}{\rho_{22}}\right]\delta u_2 + \left[\frac{e^{o}_{12} - \varphi}{R_1} + \frac{e^{o}_{12} + \varphi}{R_2}\right]\delta u_3 \\ + \left(e^{o}_{12} - \varphi\right)\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \xi_1} + \left(1 + e^{o}_{11}\right)\frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} + \left(1 + e^{o}_{22}\right)\frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} \\ + \left(e^{o}_{12} + \varphi\right)\frac{1}{A_2}\frac{\partial \delta u_2}{\partial \xi_2} - \varphi_2\frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1} - \varphi_1\frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2}\end{aligned} \tag{24c}$$

and

$$\delta\chi_{11}^{o} = \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{R_1}\right)\delta u_1 + \frac{1}{R_1}\frac{1}{A_1}\frac{\partial\delta u_1}{\partial\xi_1} - \frac{1}{\rho_{11}}\frac{\delta u_2}{R_2} + \frac{1}{\rho_{11}}\frac{1}{A_2}\frac{\partial\delta u_3}{\partial\xi_2} - \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_1}\frac{\partial\delta u_3}{\partial\xi_1}\right) \quad (25a)$$

$$\delta\chi_{22}^{o} = \frac{1}{\rho_{22}}\frac{\delta u_1}{R_1} + \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{R_2}\right)\delta u_2 + \frac{1}{R_2}\frac{1}{A_2}\frac{\partial\delta u_2}{\partial\xi_2} - \frac{1}{\rho_{22}}\frac{1}{A_1}\frac{\partial\delta u_3}{\partial\xi_1} - \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_2}\frac{\partial\delta u_3}{\partial\xi_2}\right) \quad (25b)$$

$$\begin{aligned}\delta 2\chi_{12}^{o} = & \left[\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{R_1}\right) + \frac{1}{2\rho_{11}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]\delta u_1 + \left[\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{R_2}\right) - \frac{1}{2\rho_{22}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]\delta u_2 \\ & + \frac{1}{2}\left(\frac{3}{R_1} - \frac{1}{R_2}\right)\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\xi_2} + \frac{1}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right)\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\xi_1} - \frac{1}{\rho_{11}}\left(\frac{1}{A_1}\frac{\partial\delta u_3}{\partial\xi_1}\right) \\ & + \frac{1}{\rho_{22}}\left(\frac{1}{A_2}\frac{\partial\delta u_3}{\partial\xi_2}\right) - \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_1}\frac{\partial\delta u_3}{\partial\xi_1}\right) - \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_2}\frac{\partial\delta u_3}{\partial\xi_2}\right)\end{aligned} \quad (25c)$$

Equation (22a) is now expressed in matrix form as

$$\{\boldsymbol{\delta\varepsilon^{o}}\} = [\mathbf{d_0}]\{\boldsymbol{\delta u}\} + [\mathbf{d_1}]\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\{\boldsymbol{\delta u}\} + [\mathbf{d_2}]\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\{\boldsymbol{\delta u}\} \quad (26)$$

where

$$\{\boldsymbol{\delta u}\} = \begin{Bmatrix}\delta u_1 \\ \delta u_2 \\ \delta u_3\end{Bmatrix} \quad (27a)$$

$$[\mathbf{d_0}] = \begin{bmatrix} \frac{\varphi_1}{R_1} + \frac{e_{12}^{o} + \varphi}{\rho_{11}} & -\frac{1 + e_{11}^{o}}{\rho_{11}} & \frac{1 + e_{11}^{o}}{R_1} \\ \frac{1 + e_{22}^{o}}{\rho_{22}} & \frac{\varphi_2}{R_2} - \frac{e_{12}^{o} - \varphi}{\rho_{22}} & \frac{1 + e_{22}^{o}}{R_2} \\ \frac{\varphi_2}{R_1} + \frac{1 + e_{22}^{o}}{\rho_{11}} + \frac{e_{12}^{o} + \varphi}{\rho_{22}} & \frac{\varphi_1}{R_2} - \frac{e_{12}^{o} - \varphi}{\rho_{11}} - \frac{1 + e_{11}^{o}}{\rho_{22}} & \frac{e_{12}^{o} - \varphi}{R_1} + \frac{e_{12}^{o} + \varphi}{R_2} \end{bmatrix} \quad (27b)$$

$$[\mathbf{d_1}] = \begin{bmatrix} 1 + e_{11}^{o} & e_{12}^{o} + \varphi & -\varphi_1 \\ 0 & 0 & 0 \\ e_{12}^{o} - \varphi & 1 + e_{22}^{o} & -\varphi_2 \end{bmatrix} \quad (27c)$$

$$
[\mathbf{d_2}] = \begin{bmatrix} 0 & 0 & 0 \\ e^{o}_{12} - \varphi & 1 + e^{o}_{22} & -\varphi_2 \\ 1 + e^{o}_{11} & e^{o}_{12} + \varphi & -\varphi_1 \end{bmatrix} \tag{27d}
$$

Likewise, equation (22b) is now expressed in matrix form as

$$
\begin{aligned}
\{\boldsymbol{\delta\chi^{o}}\} = [\mathbf{k_0}]\{\mathbf{\delta u}\} + [\mathbf{k_1}]\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\{\mathbf{\delta u}\} + [\mathbf{k_2}]\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\{\mathbf{\delta u}\} \\
- [\mathbf{k_{11}}]\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\{\mathbf{\delta u}\}\right) - [\mathbf{k_{12}}]\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\{\mathbf{\delta u}\}\right) \\
- [\mathbf{k_{12}}]\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\{\mathbf{\delta u}\}\right) - [\mathbf{k_{22}}]\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\{\mathbf{\delta u}\}\right)
\end{aligned} \tag{28}
$$

where

$$
[\mathbf{k_0}] = \begin{bmatrix} \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{R_1}\right) & -\frac{1}{\rho_{11}R_2} & 0 \\ \frac{1}{\rho_{22}R_1} & \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{R_2}\right) & 0 \\ \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{1}{R_1}\right) + \frac{1}{2\rho_{11}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{1}{R_2}\right) - \frac{1}{2\rho_{22}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & 0 \end{bmatrix} \tag{29a}
$$

$$
[\mathbf{k_1}] = \begin{bmatrix} \frac{1}{R_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho_{22}} \\ 0 & \frac{1}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right) & -\frac{1}{\rho_{11}} \end{bmatrix} \tag{29b}
$$

$$
[\mathbf{k}_2] = \begin{bmatrix} 0 & 0 & \dfrac{1}{\rho_{11}} \\ 0 & \dfrac{1}{R_2} & 0 \\ \dfrac{1}{2}\left(\dfrac{3}{R_1} - \dfrac{1}{R_2}\right) & 0 & \dfrac{1}{\rho_{22}} \end{bmatrix} \tag{29c}
$$

$$
[\mathbf{k}_{11}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29d}
$$

$$
[\mathbf{k}_{12}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29e}
$$

$$
[\mathbf{k}_{22}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{29f}
$$

Equilibrium Equations and Boundary Conditions

Equilibrium equations and boundary conditions that are work conjugate to the strains appearing in equations (15) are obtained by applying the principle of virtual work. The statement of this principle for the shells considered herein is given by

$$
\iint_A \delta\mathcal{W}_I \, A_1 A_2 d\xi_1 d\xi_2 = \iint_A \delta\mathcal{W}_E \, A_1 A_2 d\xi_1 d\xi_2 + \int_{\partial A} \delta\mathcal{W}_E^B \, ds \tag{30}
$$

where $\delta\mathcal{W}_I$ is the virtual work of the internal stresses and $\delta\mathcal{W}_E$ is the virtual work of the external surface tractions acting at each point of the shell reference surface A depicted in figure 1. The symbol $\delta\mathcal{W}_E^B$ represents the virtual work of the external tractions acting on the boundary curve ∂A that encloses the region A, as shown in figure 1. The specific form of the internal virtual work needed to obtain the equilibrium equations is obtained by substituting equations (26) and (28) into equation (19b). The result of this substitution yields

$$
\begin{aligned}
\delta\mathcal{W}_I = &\left(\{\mathcal{N}\}^T[\mathbf{d_0}] + \{\mathcal{M}\}^T[\mathbf{k_0}]\right)\{\boldsymbol{\delta}\mathbf{u}\} \\
&+ \left(\{\mathcal{N}\}^T[\mathbf{d_1}] + \{\mathcal{M}\}^T[\mathbf{k_1}]\right)\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1} + \left(\{\mathcal{N}\}^T[\mathbf{d_2}] + \{\mathcal{M}\}^T[\mathbf{k_2}]\right)\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2} \\
&- \{\mathcal{M}\}^T[\mathbf{k_{11}}]\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right) - \{\mathcal{M}\}^T[\mathbf{k_{12}}]\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right) \\
&- \{\mathcal{M}\}^T[\mathbf{k_{12}}]\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right) - \{\mathcal{M}\}^T[\mathbf{k_{22}}]\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right) \\
&+ \{\mathcal{F}_1\}^T\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta\gamma}^o\}}{\partial\xi_1} + \{\mathcal{F}_2\}^T\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta\gamma}^o\}}{\partial\xi_2} + \{\mathcal{Q}\}^T\{\boldsymbol{\delta\gamma}^o\}
\end{aligned}
\qquad (31)
$$

The pointwise external virtual work of the tangential surface tractions q_1 and q_2 and the normal surface traction q_3 is

$$
\boldsymbol{\delta}\mathcal{W}_E = q_1\delta u_1 + q_2\delta u_2 + q_3\delta u_3 = \{\mathbf{q}\}^T\{\boldsymbol{\delta}\mathbf{u}\} \qquad (32a)
$$

where

$$
\{\mathbf{q}\} \equiv \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} q_1^D + q_3^L\varphi_1 \\ q_2^D + q_3^L\varphi_2 \\ q_3^D + q_3^L\left(e_{11}^o + e_{22}^o\right) + \dfrac{\partial q_3^L}{\partial\xi_1}u_1 + \dfrac{\partial q_3^L}{\partial\xi_2}u_2 + \dfrac{\partial q_3^L}{\partial\xi_3}u_3 \end{Bmatrix} \qquad (32b)
$$

The surface tractions q_1, q_2, and q_3 are defined to be positive-valued in the positive ξ_1-, ξ_2-, and ξ_3-coordinate directions, respectively, as shown in figure 2, and include the effects of a live normal-pressure field (see Appendix A), denoted by the superscript "L" and dead surface tractions, denoted by the superscript "D." The boundary integral in equation (30) represents the virtual work of forces per unit length that are applied to the boundary ∂A of the region A, and it is implied that the integrand is evaluated on the boundary. The symbol ds denotes the boundary differential arc-length coordinate, which is traversed in accordance with the surface divergence theorem of Calculus. For many practical cases, the domain of the surface A is given by $a_1 \le \xi_1 \le b_1$ and $a_2 \le \xi_2 \le b_2$, and the boundary curve ∂A consists of four smooth arcs given by the constant values of the coordinates ξ_1 and ξ_2, as depicted in figure 2. The general form of the boundary integral is given by

$$\int_{\partial A} \delta\mathcal{W}_E^B \, ds = \int_{\partial A} \left[\delta\mathcal{W}_{E1}^B\left(\hat{N} \bullet \hat{a}_1\right) + \delta\mathcal{W}_{E2}^B\left(\hat{N} \bullet \hat{a}_2\right)\right]ds \tag{33a}$$

where

$$\delta\mathcal{W}_{E1}^B = N_1\delta u_1 + S_1\delta u_2 + Q_1\delta u_3 + M_1\delta\varphi_1 + M_{12}\delta\varphi_2 \tag{33b}$$

$$\delta\mathcal{W}_{E2}^B = S_2\delta u_1 + N_2\delta u_2 + Q_2\delta u_3 + M_{21}\delta\varphi_1 + M_2\delta\varphi_2 \tag{33c}$$

In equations (33); N_1, S_1, Q_1 , N_2, S_2, and Q_2 are the ξ_1 and ξ_2 components of the external forces per unit length that are applied normal, tangential, and transverse to the given edge, respectively, as shown in figure 2. Likewise, M_1 and M_2 are the components of the moment per unit length with an axis of rotation that is parallel to the given edge, at the given point of the boundary. In addition, M_{12} and M_{21} are applied twisting moment per unit length with an axis of rotation that is perpendicular to the given edge, at the given point of the boundary.

The equilibrium equations and boundary conditions are obtained by applying "integration-by-parts" formulas, obtained by specialization of the surface divergence theorem, to the first integral in equation (30). For two arbitrary differentiable functions $f(\xi_1,\xi_2)$ and $g(\xi_1,\xi_2)$, the integration-by-parts formulas are given in general form by

$$\iint_A \frac{\partial f}{\partial \xi_1}(g)\, d\xi_1 d\xi_2 = -\iint_A (f)\frac{\partial g}{\partial \xi_1}\, d\xi_1 d\xi_2 + \int_{\partial A} \frac{fg}{A_2}\left(\hat{N} \bullet \hat{a}_1\right) ds \tag{34a}$$

$$\iint_A \frac{\partial f}{\partial \xi_2}(g)\, d\xi_1 d\xi_2 = -\iint_A (f)\frac{\partial g}{\partial \xi_2}\, d\xi_1 d\xi_2 + \int_{\partial A} \frac{fg}{A_1}\left(\hat{N} \bullet \hat{a}_2\right) ds \tag{34b}$$

where $\hat{N}$ is the outward unit-magnitude vector field that is perpendicular to points of ∂A, and that lies in the corresponding reference-surface tangent plane. In addition, $\hat{a}_1$ and $\hat{a}_2$ are unit-magnitude vector fields that are tangent to the ξ_1 and ξ_2 coordinate curves, respectively, at every point of A and ∂A, as shown in figure 1.

The integration-by-parts formulas are easily extended to a useful vector form by noting that the product $\{\mathbf{v}\}^T\{\mathbf{w}\}$ represent a linear combination of scalar pairs. Thus, the vector forms of equations (34) are given by

$$\iint_A \{\mathbf{f}\}^{T}\frac{\partial\{\mathbf{g}\}}{\partial\xi_1}\, d\xi_1 d\xi_2 = -\iint_A \frac{\partial\{\mathbf{f}\}^{T}}{\partial\xi_1}\{\mathbf{g}\}\, d\xi_1 d\xi_2 + \int_{\partial A} \frac{\{\mathbf{f}\}^{T}\{\mathbf{g}\}}{A_2}\left(\hat{N}\bullet\hat{a}_1\right) ds \tag{35a}$$

$$\iint_A \{\mathbf{f}\}^{T}\frac{\partial\{\mathbf{g}\}}{\partial\xi_2}\, d\xi_1 d\xi_2 = -\iint_A \frac{\partial\{\mathbf{f}\}^{T}}{\partial\xi_2}\{\mathbf{g}\}\, d\xi_1 d\xi_2 + \int_{\partial A} \frac{\{\mathbf{f}\}^{T}\{\mathbf{g}\}}{A_1}\left(\hat{N}\bullet\hat{a}_2\right) ds \tag{35b}$$

Applying these equations to the left-hand side of equation (30), and using equation (31) for $\delta\mathcal{W}_I$, gives the following results:

$$\begin{aligned}
&\iint_A \left(\{\mathcal{N}\}^{T}[\mathbf{d}_1] + \{\mathcal{M}\}^{T}[\mathbf{k}_1]\right)\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\, A_1A_2 d\xi_1 d\xi_2 = \\
&\quad -\iint_A \frac{\partial}{\partial\xi_1}\left(A_2\left(\{\mathcal{N}\}^{T}[\mathbf{d}_1] + \{\mathcal{M}\}^{T}[\mathbf{k}_1]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\} d\xi_1 d\xi_2 \\
&\quad + \int_{\partial A} \left(\{\mathcal{N}\}^{T}[\mathbf{d}_1] + \{\mathcal{M}\}^{T}[\mathbf{k}_1]\right)\{\boldsymbol{\delta}\mathbf{u}\}\left(\hat{N}\bullet\hat{a}_1\right) ds
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\iint_A \left(\{\mathcal{N}\}^{T}[\mathbf{d}_2] + \{\mathcal{M}\}^{T}[\mathbf{k}_2]\right)\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\, A_1A_2 d\xi_1 d\xi_2 = \\
&\quad -\iint_A \frac{\partial}{\partial\xi_2}\left(A_1\left(\{\mathcal{N}\}^{T}[\mathbf{d}_2] + \{\mathcal{M}\}^{T}[\mathbf{k}_2]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\} d\xi_1 d\xi_2 \\
&\quad + \int_{\partial A} \left(\{\mathcal{N}\}^{T}[\mathbf{d}_2] + \{\mathcal{M}\}^{T}[\mathbf{k}_2]\right)\{\boldsymbol{\delta}\mathbf{u}\}\left(\hat{N}\bullet\hat{a}_2\right) ds
\end{aligned} \tag{37}$$

$$\begin{aligned}
&-\iint_A A_2\{\mathcal{M}\}^{T}[\mathbf{k}_{11}]\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right) d\xi_1 d\xi_2 = \\
&\quad -\iint_A \frac{\partial}{\partial\xi_1}\left(\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^{T}[\mathbf{k}_{11}]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\}\, d\xi_1 d\xi_2 \\
&\quad + \int_{\partial A} \left(\frac{1}{A_1A_2}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^{T}[\mathbf{k}_{11}]\right)\{\boldsymbol{\delta}\mathbf{u}\} - \{\mathcal{M}\}^{T}[\mathbf{k}_{11}]\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right)\left(\hat{N}\bullet\hat{a}_1\right) ds
\end{aligned} \tag{38}$$

$$
\begin{aligned}
-\iint_A & A_1\{\mathcal{M}\}^T[\mathbf{k}_{22}]\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right)d\xi_1 d\xi_2 = \\
& -\iint_A \frac{\partial}{\partial\xi_2}\left(\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T[\mathbf{k}_{22}]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\}\, d\xi_1 d\xi_2 \\
& + \int_{\partial A}\left(\frac{1}{A_1A_2}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T[\mathbf{k}_{22}]\right)\{\boldsymbol{\delta}\mathbf{u}\} - \{\mathcal{M}\}^T[\mathbf{k}_{22}]\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right)\left(\hat{N}\bullet\hat{a}_2\right)ds
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
-\iint_A & A_2\{\mathcal{M}\}^T[\mathbf{k}_{12}]\frac{\partial}{\partial\xi_1}\left(\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right)d\xi_1 d\xi_2 = \\
& -\iint_A \frac{\partial}{\partial\xi_2}\left(\frac{1}{A_2}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^T[\mathbf{k}_{12}]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\}\, d\xi_1 d\xi_2 \\
& + \int_{\partial A}\frac{1}{A_1A_2}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta}\mathbf{u}\}\left(\hat{N}\bullet\hat{a}_2\right)ds \\
& - \int_{\partial A}\{\mathcal{M}\}^T[\mathbf{k}_{12}]\left(\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}\right)\left(\hat{N}\bullet\hat{a}_1\right)ds
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
-\iint_A & A_1\{\mathcal{M}\}^T[\mathbf{k}_{12}]\frac{\partial}{\partial\xi_2}\left(\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right)d\xi_1 d\xi_2 = \\
& -\iint_A \frac{\partial}{\partial\xi_1}\left(\frac{1}{A_1}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T[\mathbf{k}_{12}]\right)\right)\{\boldsymbol{\delta}\mathbf{u}\}\, d\xi_1 d\xi_2 \\
& + \int_{\partial A}\frac{1}{A_1A_2}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta}\mathbf{u}\}\left(\hat{N}\bullet\hat{a}_1\right)ds \\
& - \int_{\partial A}\{\mathcal{M}\}^T[\mathbf{k}_{12}]\left(\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_1}\right)\left(\hat{N}\bullet\hat{a}_2\right)ds
\end{aligned}
\tag{41}
$$

$$\iint_A A_2\{\mathcal{F}_1\}^T \frac{\partial\{\boldsymbol{\delta\gamma}^o\}}{\partial\xi_1}\, d\xi_1 d\xi_2 =$$
$$-\iint_A \frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{F}_1\}^T\right)\{\boldsymbol{\delta\gamma}^o\}\, d\xi_1 d\xi_2 + \int_{\partial A} \{\mathcal{F}_1\}^T\{\boldsymbol{\delta\gamma}^o\}\left(\hat{N}\bullet\hat{a}_1\right) ds \tag{42}$$

$$\iint_A A_1\{\mathcal{F}_2\}^T \frac{\partial\{\boldsymbol{\delta\gamma}^o\}}{\partial\xi_2}\, d\xi_1 d\xi_2 =$$
$$-\iint_A \frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{F}_2\}^T\right)\{\boldsymbol{\delta\gamma}^o\}\, d\xi_1 d\xi_2 + \int_{\partial A} \{\mathcal{F}_2\}^T\{\boldsymbol{\delta\gamma}^o\}\left(\hat{N}\bullet\hat{a}_2\right) ds \tag{43}$$

From these equations, it follows that

$$\iint_A \boldsymbol{\delta}\mathcal{W}_I\, A_1 A_2 d\xi_1 d\xi_2 = \iint_A \left(\{\Sigma\,\mathcal{F}\}^T\{\boldsymbol{\delta u}\} + \{\Sigma\,\tilde{\mathcal{F}}\}^T\{\boldsymbol{\delta\gamma}^o\}\right) d\xi_1 d\xi_2 + \int_{\partial A} \boldsymbol{\delta}\mathcal{W}_I^B\, ds \tag{44}$$

where

$$\begin{aligned}\{\Sigma\,\mathcal{F}\}^T &= A_1 A_2\left(\{\mathcal{N}\}^T[\mathbf{d_0}] + \{\mathcal{M}\}^T[\mathbf{k_0}]\right)\\ &- \frac{\partial}{\partial\xi_1}\left(A_2\left(\{\mathcal{N}\}^T[\mathbf{d_1}] + \{\mathcal{M}\}^T[\mathbf{k_1}]\right) + \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^T\right)[\mathbf{k_{11}}] + \frac{1}{A_1}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T\right)[\mathbf{k_{12}}]\right)\\ &- \frac{\partial}{\partial\xi_2}\left(A_1\left(\{\mathcal{N}\}^T[\mathbf{d_2}] + \{\mathcal{M}\}^T[\mathbf{k_2}]\right) + \frac{1}{A_2}\frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{M}\}^T\right)[\mathbf{k_{12}}] + \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{M}\}^T\right)[\mathbf{k_{22}}]\right)\end{aligned} \tag{45a}$$

$$\{\Sigma\,\tilde{\mathcal{F}}\}^T = A_1 A_2\{\mathcal{Q}\}^T - \frac{\partial}{\partial\xi_1}\left(A_2\{\mathcal{F}_1\}^T\right) - \frac{\partial}{\partial\xi_2}\left(A_1\{\mathcal{F}_2\}^T\right) \tag{45b}$$

Next, equations (2) are used to get

$$
\begin{aligned}
\frac{1}{A_1A_2}\left\{\Sigma\,\mathcal{F}\right\}^{T} = {} & \left\{\mathcal{N}\right\}^{T}\left[\left[\mathbf{d_0}\right] - \frac{1}{\rho_{22}}\left[\mathbf{d_1}\right] + \frac{1}{\rho_{11}}\left[\mathbf{d_2}\right]\right] + \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_0}\right] \\
& - \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left[\left\{\mathcal{N}\right\}^{T}\left[\mathbf{d_1}\right]\right] - \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\left\{\mathcal{N}\right\}^{T}\left[\mathbf{d_2}\right]\right) \\
& - \frac{1}{A_1A_2}\frac{\partial}{\partial\xi_1}\left(A_2\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_1}\right] + \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(A_2\left\{\mathcal{M}\right\}^{T}\right)\left[\mathbf{k_{11}}\right] + \frac{1}{A_1}\frac{\partial}{\partial\xi_2}\left(A_1\left\{\mathcal{M}\right\}^{T}\right)\left[\mathbf{k_{12}}\right]\right) \\
& - \frac{1}{A_1A_2}\frac{\partial}{\partial\xi_2}\left(A_1\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_2}\right] + \frac{1}{A_2}\frac{\partial}{\partial\xi_1}\left(A_2\left\{\mathcal{M}\right\}^{T}\right)\left[\mathbf{k_{12}}\right] + \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(A_1\left\{\mathcal{M}\right\}^{T}\right)\left[\mathbf{k_{22}}\right]\right)
\end{aligned}
\tag{46a}
$$

$$
\frac{1}{A_1A_2}\left\{\Sigma\,\tilde{\mathcal{F}}\right\}^{T} = \left\{\mathcal{Q}\right\}^{T} + \frac{1}{\rho_{11}}\left\{\mathcal{F}_2\right\}^{T} - \frac{1}{\rho_{22}}\left\{\mathcal{F}_1\right\}^{T} - \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left\{\mathcal{F}_1\right\}^{T} - \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left\{\mathcal{F}_2\right\}^{T}
\tag{46b}
$$

The boundary integral appearing in equation (44) is given by

$$
\int_{\partial A} \boldsymbol{\delta}\mathcal{W}_{I}^{B}\,ds = \int_{\partial A}\left[\boldsymbol{\delta}\mathcal{W}_{I1}^{B}\left(\hat{N}\bullet\hat{a}_1\right) + \boldsymbol{\delta}\mathcal{W}_{I2}^{B}\left(\hat{N}\bullet\hat{a}_2\right)\right]ds
\tag{47}
$$

where

$$
\begin{aligned}
\boldsymbol{\delta}\mathcal{W}_{I1}^{B} = {} & \left[\left\{\mathcal{N}\right\}^{T}\left[\mathbf{d_1}\right] + \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_1}\right]\right]\left\{\boldsymbol{\delta}\mathbf{u}\right\} \\
& + \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\xi_1}\left(A_2\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{11}}\right]\right) + \frac{\partial}{\partial\xi_2}\left(A_1\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{12}}\right]\right)\right]\left\{\boldsymbol{\delta}\mathbf{u}\right\} \\
& - \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{11}}\right]\frac{1}{A_1}\frac{\partial\left\{\boldsymbol{\delta}\mathbf{u}\right\}}{\partial\xi_1} - \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{12}}\right]\frac{1}{A_2}\frac{\partial\left\{\boldsymbol{\delta}\mathbf{u}\right\}}{\partial\xi_2} + \left\{\mathcal{F}_1\right\}^{T}\left\{\boldsymbol{\delta\gamma}^{o}\right\}
\end{aligned}
\tag{48a}
$$

$$
\begin{aligned}
\boldsymbol{\delta}\mathcal{W}_{I2}^{B} = {} & \left(\left\{\mathcal{N}\right\}^{T}\left[\mathbf{d_2}\right] + \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_2}\right]\right)\left\{\boldsymbol{\delta}\mathbf{u}\right\} \\
& + \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\xi_1}\left(A_2\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{12}}\right]\right) + \frac{\partial}{\partial\xi_2}\left(A_1\left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{22}}\right]\right)\right]\left\{\boldsymbol{\delta}\mathbf{u}\right\} \\
& - \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{12}}\right]\frac{1}{A_1}\frac{\partial\left\{\boldsymbol{\delta}\mathbf{u}\right\}}{\partial\xi_1} - \left\{\mathcal{M}\right\}^{T}\left[\mathbf{k_{22}}\right]\frac{1}{A_2}\frac{\partial\left\{\boldsymbol{\delta}\mathbf{u}\right\}}{\partial\xi_2} + \left\{\mathcal{F}_2\right\}^{T}\left\{\boldsymbol{\delta\gamma}^{o}\right\}
\end{aligned}
\tag{48b}
$$

Next, equations (32a) and (44) are substituted into equation (30) to get

$$\iint_A \left(\left(\{\Sigma \mathcal{F}\}^T - A_1 A_2 \{\mathbf{q}\}^T \right) \{\boldsymbol{\delta}\mathbf{u}\} + \{\Sigma \tilde{\mathcal{F}}\}^T \{\boldsymbol{\delta\gamma}^o\} \right) d\xi_1 d\xi_2 + \int_{\partial A} \left(\boldsymbol{\delta}\mathcal{W}_I^B - \boldsymbol{\delta}\mathcal{W}_E^B \right) ds = 0 \quad (49)$$

as the statement of the principle of virtual work. Because $\{\boldsymbol{\delta}\mathbf{u}\}$ and $\{\boldsymbol{\delta\gamma}^o\}$ are independent virtual displacements, that are generally nonzero, the localization lemma for the Calculus of Variations yields the equilibrium equations

$$\frac{1}{A_1 A_2} \{\Sigma \mathcal{F}\}^T - \{\mathbf{q}\}^T = \{\mathbf{0}\} \quad (50a)$$

$$\{\Sigma \tilde{\mathcal{F}}\}^T = \{\mathbf{0}\} \quad (50b)$$

In order to identify the strain-displacement relations of several different theories that appear often in the literature, equation (16a) is expressed as

$$\{\boldsymbol{\varepsilon}^o\} \equiv \begin{Bmatrix} \varepsilon_{11}^o \\ \varepsilon_{22}^o \\ \gamma_{12}^o \end{Bmatrix} = \begin{Bmatrix} e_{11}^o + \frac{1}{2}\left(\varphi_1^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e_{11}^o\right)^2 + e_{12}^o\left(e_{12}^o + 2\varphi\right)\right] \\ e_{22}^o + \frac{1}{2}\left(\varphi_2^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e_{22}^o\right)^2 + e_{12}^o\left(e_{12}^o - 2\varphi\right)\right] \\ 2e_{12}^o + \varphi_1\varphi_2 + c_1\left[e_{11}^o\left(e_{12}^o - \varphi\right) + e_{22}^o\left(e_{12}^o + \varphi\right)\right] \end{Bmatrix} \quad (51)$$

and equations (4) as

$$\varphi_1(\xi_1, \xi_2) = \frac{c_3 u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial \xi_1} \quad (52a)$$

$$\varphi_2(\xi_1, \xi_2) = \frac{c_3 u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial \xi_2} \quad (52b)$$

$$\varphi(\xi_1, \xi_2) = \frac{1}{2}c_3\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}} \right) \quad (52c)$$

With this notation, equations (16a) and (8) are recovered by specifying $c_1 = c_2 = c_3 = 1$. Specifying $c_1 = 0$ and $c_2 = c_3 = 1$ gives Sanders' strain-displacement relations, and specifying $c_1 = 0$, $c_2 = 0$, and $c_3 = 1$ gives Sanders' strain-displacement relations with nonlinear rotations about the reference-surface normal neglected. In addition, specifying $c_1 = c_2 = c_3 = 0$ gives the strain-displacement relations of the Donnell-Mushtari-Vlasov theory. Accordingly,

$$\left\{\boldsymbol{\delta\varepsilon}^{o}\right\} = \begin{Bmatrix} \delta\varepsilon_{11}^{o} \\ \delta\varepsilon_{22}^{o} \\ \delta\gamma_{12}^{o} \end{Bmatrix} = \begin{Bmatrix} \left(1 + c_1 e_{11}^{o}\right)\delta e_{11}^{o} + c_1\left(e_{12}^{o} + \varphi\right)\delta e_{12}^{o} + \varphi_1 \delta\varphi_1 + \left(c_1 e_{12}^{o} + c_2\varphi\right)\delta\varphi \\ \left(1 + c_1 e_{22}^{o}\right)\delta e_{22}^{o} + c_1\left(e_{12}^{o} - \varphi\right)\delta e_{12}^{o} + \varphi_2 \delta\varphi_2 - \left(c_1 e_{12}^{o} - c_2\varphi\right)\delta\varphi \\ c_1\left(e_{12}^{o} - \varphi\right)\delta e_{11}^{o} + c_1\left(e_{12}^{o} + \varphi\right)\delta e_{22}^{o} + \left[2 + c_1\left(e_{11}^{o} + e_{22}^{o}\right)\right]\delta e_{12}^{o} \\ + \varphi_2\delta\varphi_1 + \varphi_1\delta\varphi_2 + c_1\left(e_{22}^{o} - e_{11}^{o}\right)\delta\varphi \end{Bmatrix} \tag{53}$$

where

$$\delta\varphi_1\left(\xi_1, \xi_2\right) = \frac{c_3 \delta u_1}{R_1} - \frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1} \tag{54a}$$

$$\delta\varphi_2\left(\xi_1, \xi_2\right) = \frac{c_3 \delta u_2}{R_2} - \frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2} \tag{54b}$$

$$\delta\varphi\left(\xi_1, \xi_2\right) = \frac{1}{2}c_3\left(\frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} + \frac{\delta u_1}{\rho_{11}} + \frac{\delta u_2}{\rho_{22}}\right) \tag{54c}$$

Substituting equations (23a)-(23c) and (54) into (53), the matrices in equations (26) are expressed as

$$\left[\mathbf{d_0}\right] = \begin{bmatrix} \frac{c_3\varphi_1}{R_1} + \frac{c_1\left(1 + c_3\right)e_{12}^{o} + \left(c_1 + c_2 c_3\right)\varphi}{2\rho_{11}} & \frac{c_1\left(c_3 - 1\right)e_{12}^{o} + \left(c_2 c_3 - c_1\right)\varphi}{2\rho_{22}} - \frac{1 + c_1 e_{11}^{o}}{\rho_{11}} & \frac{1 + c_1 e_{11}^{o}}{R_1} \\ \frac{c_1\left(1 - c_3\right)e_{12}^{o} + \left(c_2 c_3 - c_1\right)\varphi}{2\rho_{11}} + \frac{1 + c_1 e_{22}^{o}}{\rho_{22}} & \frac{c_3\varphi_2}{R_2} - \frac{c_1\left(1 + c_3\right)e_{12}^{o} - \left(c_1 + c_2 c_3\right)\varphi}{2\rho_{22}} & \frac{1 + c_1 e_{22}^{o}}{R_2} \\ \frac{c_3\varphi_2}{R_1} + \frac{2 + c_1\left(1 - c_3\right)e_{11}^{o} + c_1\left(1 + c_3\right)e_{22}^{o}}{2\rho_{11}} + \frac{c_1\left(e_{12}^{o} + \varphi\right)}{\rho_{22}} & \frac{c_3\varphi_1}{R_2} - \frac{2 + c_1\left(1 + c_3\right)e_{11}^{o} + c_1\left(1 - c_3\right)e_{22}^{o}}{2\rho_{22}} - \frac{c_1\left(e_{12}^{o} - \varphi\right)}{\rho_{11}} & c_1\left(\frac{e_{12}^{o} - \varphi}{R_1} + \frac{e_{12}^{o} + \varphi}{R_2}\right) \end{bmatrix} \tag{55a}$$

$$[\mathbf{d_1}] = \begin{bmatrix} 1 + c_1 e^{o}_{11} & \frac{c_1}{2}\left(e^{o}_{12} + \varphi\right) + \frac{c_3}{2}\left(c_1 e^{o}_{12} + c_2\varphi\right) & -\varphi_1 \\ 0 & \frac{c_1}{2}\left(e^{o}_{12} - \varphi\right) - \frac{c_3}{2}\left(c_1 e^{o}_{12} - c_2\varphi\right) & 0 \\ c_1\left(e^{o}_{12} - \varphi\right) & 1 + \frac{c_1}{2}\left(1 - c_3\right)e^{o}_{11} + \frac{c_1}{2}\left(1 + c_3\right)e^{o}_{22} & -\varphi_2 \end{bmatrix} \tag{55b}$$

$$[\mathbf{d_2}] = \begin{bmatrix} \frac{c_1}{2}\left(e^{o}_{12} + \varphi\right) - \frac{c_3}{2}\left(c_1 e^{o}_{12} + c_2\varphi\right) & 0 & 0 \\ \frac{c_1}{2}\left(e^{o}_{12} - \varphi\right) + \frac{c_3}{2}\left(c_1 e^{o}_{12} - c_2\varphi\right) & 1 + c_1 e^{o}_{22} & -\varphi_2 \\ 1 + \frac{c_1}{2}\left(1 + c_3\right)e^{o}_{11} + \frac{c_1}{2}\left(1 - c_3\right)e^{o}_{22} & c_1\left(e^{o}_{12} + \varphi\right) & -\varphi_1 \end{bmatrix} \tag{55c}$$

Moreover,

$$[\mathbf{d_0}] - \frac{1}{\rho_{22}}[\mathbf{d_1}] + \frac{1}{\rho_{11}}[\mathbf{d_2}] = \tag{56}$$

$$\begin{bmatrix} \frac{c_3\varphi_1}{R_1} + \frac{c_1\left(e^{o}_{12} + \varphi\right)}{\rho_{11}} - \frac{1 + c_1 e^{o}_{11}}{\rho_{22}} & -\frac{c_1\left(e^{o}_{12} + \varphi\right)}{\rho_{22}} - \frac{1 + c_1 e^{o}_{11}}{\rho_{11}} & \frac{1 + c_1 e^{o}_{11}}{R_1} + \frac{\varphi_1}{\rho_{22}} \\ \frac{c_1\left(e^{o}_{12} - \varphi\right)}{\rho_{11}} + \frac{1 + c_1 e^{o}_{22}}{\rho_{22}} & \frac{c_3\varphi_2}{R_2} - \frac{c_1\left(e^{o}_{12} - \varphi\right)}{\rho_{22}} + \frac{1 + c_1 e^{o}_{22}}{\rho_{11}} & \frac{1 + c_1 e^{o}_{22}}{R_2} - \frac{\varphi_2}{\rho_{11}} \\ \frac{c_3\varphi_2}{R_1} + \frac{2 + c_1\left(e^{o}_{11} + e^{o}_{22}\right)}{\rho_{11}} + \frac{2c_1\varphi}{\rho_{22}} & \frac{c_3\varphi_1}{R_2} - \frac{2 + c_1\left(e^{o}_{11} + e^{o}_{22}\right)}{\rho_{22}} + \frac{2c_1\varphi}{\rho_{11}} & c_1\left(\frac{e^{o}_{12} - \varphi}{R_1} + \frac{e^{o}_{12} + \varphi}{R_2}\right) + \frac{\varphi_2}{\rho_{22}} - \frac{\varphi_1}{\rho_{11}} \end{bmatrix}$$

In addition, substituting equations (54) in to equation (22b) yields the revised matrices

$$[\mathbf{k_0}] = c_3 \begin{bmatrix} \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{R_1}\right) & -\frac{1}{\rho_{11}R_2} & 0 \\ \frac{1}{\rho_{22}R_1} & \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{R_2}\right) & 0 \\ \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\frac{1}{R_1}\right) + \frac{1}{2\rho_{11}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\frac{1}{R_2}\right) - \frac{1}{2\rho_{22}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & 0 \end{bmatrix} \tag{57a}$$

$$[\mathbf{k}_1] = \begin{bmatrix} \frac{c_3}{R_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho_{22}} \\ 0 & \frac{c_3}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right) & -\frac{1}{\rho_{11}} \end{bmatrix} \tag{57b}$$

$$[\mathbf{k}_2] = \begin{bmatrix} 0 & 0 & \frac{1}{\rho_{11}} \\ 0 & \frac{c_3}{R_2} & 0 \\ \frac{c_3}{2}\left(\frac{3}{R_1} - \frac{1}{R_2}\right) & 0 & \frac{1}{\rho_{22}} \end{bmatrix} \tag{57c}$$

that appear in equation (28). Equations (29d)-(29f) remain unaltered. By applying these revised matrices to equations (46) and (50), the equilibrium equations are found to be

$$\frac{1}{A_1}\frac{\partial \mathcal{N}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{12}}{\partial \xi_2} - \frac{2\mathcal{N}_{12}}{\rho_{11}} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{22}} + \frac{c_3\tilde{Q}_{13}}{R_1} + \frac{c_3}{2A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{M}_{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] + \mathcal{P}_1 + q_1 = 0 \tag{58a}$$

$$\frac{1}{A_1}\frac{\partial \mathcal{N}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{22}}{\partial \xi_2} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{11}} + \frac{2\mathcal{N}_{12}}{\rho_{22}} + \frac{c_3\tilde{Q}_{23}}{R_2} + \frac{c_3}{2A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{M}_{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\right] + \mathcal{P}_2 + q_2 = 0 \tag{58b}$$

$$\frac{1}{A_1}\frac{\partial \tilde{Q}_{13}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \tilde{Q}_{23}}{\partial \xi_2} + \frac{\tilde{Q}_{13}}{\rho_{22}} - \frac{\tilde{Q}_{23}}{\rho_{11}} - \frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2} + \mathcal{P}_3 + q_3 = 0 \tag{58c}$$

$$\mathcal{Q}_{13} + \frac{\mathcal{F}_{21}}{\rho_{11}} - \frac{\mathcal{F}_{11}}{\rho_{22}} - \frac{1}{A_1}\frac{\partial \mathcal{F}_{11}}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \mathcal{F}_{21}}{\partial \xi_2} = 0 \tag{58d}$$

$$\mathcal{Q}_{23} + \frac{\mathcal{F}_{22}}{\rho_{11}} - \frac{\mathcal{F}_{12}}{\rho_{22}} - \frac{1}{A_1}\frac{\partial \mathcal{F}_{12}}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \mathcal{F}_{22}}{\partial \xi_2} = 0 \tag{58e}$$

where

$$\tilde{Q}_{13} \equiv \frac{1}{A_1}\frac{\partial \mathcal{M}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial \xi_2} + \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{\rho_{22}} - \frac{2\mathcal{M}_{12}}{\rho_{11}} \tag{59a}$$

$$\tilde{Q}_{23} \equiv \frac{1}{A_1}\frac{\partial \mathcal{M}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{22}}{\partial \xi_2} + \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{\rho_{11}} + \frac{2\mathcal{M}_{12}}{\rho_{22}} \tag{59b}$$

$$\begin{aligned}\mathcal{P}_1 = &- \frac{c_3}{R_1}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{c_2}{2}\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left[\varphi\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\right] + \frac{c_1}{A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{N}_{11}e^{o}_{11} + \mathcal{N}_{12}\left(e^{o}_{12} - \varphi\right)\right] \\ &- \frac{c_1}{\rho_{11}}\left[\mathcal{N}_{11}\left(e^{o}_{12} + \varphi\right) + \mathcal{N}_{22}\left(e^{o}_{12} - \varphi\right) + \mathcal{N}_{12}\left(e^{o}_{11} + e^{o}_{22}\right)\right] - \frac{c_1}{\rho_{22}}\left[\mathcal{N}_{22}e^{o}_{22} - \mathcal{N}_{11}e^{o}_{11} + 2\mathcal{N}_{12}\varphi\right] \\ &+ \frac{c_1}{2}\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left[\left(\mathcal{N}_{11} - \mathcal{N}_{22}\right)\varphi + 2\left(\mathcal{N}_{22}e^{o}_{12} + \mathcal{N}_{12}e^{o}_{11}\right)\right]\end{aligned} \tag{59c}$$

$$\begin{aligned}\mathcal{P}_2 = &- \frac{c_3}{R_2}\left[\mathcal{N}_{22}\varphi_2 + \mathcal{N}_{12}\varphi_1\right] - \frac{c_2}{2A_1}\frac{\partial}{\partial \xi_1}\left[\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi\right] + \frac{c_1}{A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{N}_{22}e^{o}_{22} + \mathcal{N}_{12}\left(e^{o}_{12} + \varphi\right)\right] \\ &+ \frac{c_1}{\rho_{22}}\left[\mathcal{N}_{11}\left(e^{o}_{12} + \varphi\right) + \mathcal{N}_{22}\left(e^{o}_{12} - \varphi\right) + \mathcal{N}_{12}\left(e^{o}_{11} + e^{o}_{22}\right)\right] - \frac{c_1}{\rho_{11}}\left[\mathcal{N}_{22}e^{o}_{22} - \mathcal{N}_{11}e^{o}_{11} + 2\mathcal{N}_{12}\varphi\right] \\ &+ \frac{c_1}{2A_1}\frac{\partial}{\partial \xi_1}\left[\left(\mathcal{N}_{11} - \mathcal{N}_{22}\right)\varphi + 2\left(\mathcal{N}_{11}e^{o}_{12} + \mathcal{N}_{12}e^{o}_{22}\right)\right]\end{aligned} \tag{59d}$$

$$\begin{aligned}\mathcal{P}_3 = &- \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{1}{\rho_{22}}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{N}_{12}\varphi_1 + \mathcal{N}_{22}\varphi_2\right] \\ &+ \frac{1}{\rho_{11}}\left[\mathcal{N}_{12}\varphi_1 + \mathcal{N}_{22}\varphi_2\right] - \frac{c_1}{R_1}\left[\mathcal{N}_{11}e^{o}_{11} + \mathcal{N}_{12}\left(e^{o}_{12} - \varphi\right)\right] - \frac{c_1}{R_2}\left[\mathcal{N}_{22}e^{o}_{22} + \mathcal{N}_{12}\left(e^{o}_{12} + \varphi\right)\right]\end{aligned} \tag{59e}$$

Before the boundary conditions can be obtained, the boundary integral given by equation (43) must be reduced further. In particular, by noting that

$$\left(\hat{N} \bullet \hat{a}_1\right)ds = A_2\, d\xi_2 \tag{60a}$$

$$\left(\hat{N} \bullet \hat{a}_2\right)ds = -\, A_1\, d\xi_1 \tag{60b}$$

it is seen that the integrals

$$\int_{\partial A} \{\mathcal{M}\}^{T}\left[\mathbf{k}_{12}\right]\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial \xi_2}\left(\hat{N} \bullet \hat{a}_1\right)ds \tag{61a}$$

$$\int_{\partial A} \{\mathcal{M}\}^{T}\left[\mathbf{k}_{12}\right]\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial \xi_1}\left(\hat{N} \bullet \hat{a}_2\right)ds \tag{61b}$$

can be integrated by parts further, by using the product rule of differentiation, to get

$$\int_{\partial A} \{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\left[\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta u}\}}{\partial\xi_2}\left(\hat{N}\bullet\hat{a}_1\right)+\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta u}\}}{\partial\xi_1}\left(\hat{N}\bullet\hat{a}_2\right)\right]ds =$$

$$-\int_{\partial A}\left[\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\left(\hat{N}\bullet\hat{a}_2\right)ds+\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\left(\hat{N}\bullet\hat{a}_1\right)ds\right]\{\boldsymbol{\delta u}\} \qquad (62)$$

$$+\int_{\partial A}\frac{\partial}{\partial\xi_2}\left[\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta u}\}\right]d\xi_2-\int_{\partial A}\frac{\partial}{\partial\xi_1}\left[\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta u}\}\right]d\xi_1$$

Using this result, equation (47) is expressed as

$$\int_{\partial A}\boldsymbol{\delta}\mathcal{W}_1^B\,ds=\int_{\partial A}\left[\boldsymbol{\delta}\widehat{\mathcal{W}}_{11}^B\left(\hat{N}\bullet\hat{a}_1\right)+\boldsymbol{\delta}\widehat{\mathcal{W}}_{12}^B\left(\hat{N}\bullet\hat{a}_2\right)\right]ds+\left\langle\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\{\boldsymbol{\delta u}\}\right\rangle_{\partial A} \qquad (63)$$

where

$$\left\langle\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\{\boldsymbol{\delta u}\}\right\rangle_{\partial A}\equiv\int_{\partial A}\frac{\partial}{\partial\xi_2}\left[\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta u}\}\right]d\xi_2-\int_{\partial A}\frac{\partial}{\partial\xi_1}\left[\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\{\boldsymbol{\delta u}\}\right]d\xi_1 \qquad (64a)$$

$$\boldsymbol{\delta}\widehat{\mathcal{W}}_{11}^B=\left[\{\boldsymbol{\mathcal{N}}\}^T[\mathbf{d}_1]+\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_1]+\frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left(\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\right]\{\boldsymbol{\delta u}\}$$

$$+\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\xi_1}\left(A_2\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{11}]\right)+\frac{\partial}{\partial\xi_2}\left(A_1\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{12}]\right)\right]\{\boldsymbol{\delta u}\} \qquad (64b)$$

$$-\{\boldsymbol{\mathcal{M}}\}^T[\mathbf{k}_{11}]\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta u}\}}{\partial\xi_1}+\{\mathcal{F}_1\}^T\{\boldsymbol{\delta\gamma}^o\}$$

$$\delta\widehat{\mathcal{W}}_{I2}^{B} = \left[\{\boldsymbol{\mathcal{N}}\}^{T}[\mathbf{d}_2] + \{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_2] + \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left(\{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{12}] \right) \right]\{\boldsymbol{\delta u}\}$$
$$+ \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\xi_1}\left(A_2\{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{12}] \right) + \frac{\partial}{\partial\xi_2}\left(A_1\{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{22}] \right) \right]\{\boldsymbol{\delta u}\} \tag{64c}$$
$$- \{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{22}]\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta u}\}}{\partial\xi_2} + \{\boldsymbol{\mathcal{F}}_2\}^{T}\{\boldsymbol{\delta\gamma}^{o}\}$$

On an edge given by ξ_1 = constant; $d\xi_1 = 0$, $(\hat{N} \bullet \hat{a}_1) = 1$, and $(\hat{N} \bullet \hat{a}_2) = 0$. For this case,

$$\int_{\partial A} \delta\mathcal{W}_{I}^{B}\, ds = \int_{a_2}^{b_2} \left[\delta\widehat{\mathcal{W}}_{I1}^{B} A_2 \right]_{\xi_1 = \text{constant}} d\xi_2 + \left(\left\langle \{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{12}]\{\boldsymbol{\delta u}\} \right\rangle_{\xi_1 = \text{constant}} \right)_{a_2}^{b_2} \tag{65a}$$

where

$$\left(\left\langle \{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{12}]\{\boldsymbol{\delta u}\} \right\rangle_{\xi_1 = \text{constant}} \right)_{a_2}^{b_2} = \left(\left[\{\boldsymbol{\mathcal{M}}\}^{T}[\mathbf{k}_{12}]\{\boldsymbol{\delta u}\} \right]_{\xi_1 = \text{constant}} \right)_{a_2}^{b_2} \tag{65b}$$

Likewise, equation (33a) has the form

$$\int_{\partial A} \delta\mathcal{W}_{E}^{B}\, ds = \int_{a_2}^{b_2} \left[\delta\overline{\overline{\mathcal{W}}}_{E1}^{B} A_2 \right]_{\xi_1 = \text{constant}} d\xi_2 \tag{66a}$$

with

$$\delta\overline{\overline{\mathcal{W}}}_{E1}^{B} = \left[N_1(\xi_2) + \frac{c_3M_1(\xi_2)}{R_1} \right]\delta u_1 + \left[S_1(\xi_2) + \frac{c_3M_{12}(\xi_2)}{R_2} \right]\delta u_2 + Q_1(\xi_2)\delta u_3$$
$$- \frac{M_1(\xi_2)}{A_1}\frac{\partial\delta u_3}{\partial\xi_1} - \frac{M_{12}(\xi_2)}{A_2}\frac{\partial\delta u_3}{\partial\xi_2} \tag{66b}$$

where equations (54) have been used for the virtual rotations. Noting that

$$\int_{a_2}^{b_2} \left[M_{12}(\xi_2)\frac{\partial\delta u_3}{\partial\xi_2} \right]_{\xi_1 = \text{constant}} d\xi_2 = -\int_{a_2}^{b_2} \left[\frac{dM_{12}(\xi_2)}{d\xi_2}\delta u_3 \right]_{\xi_1 = \text{constant}} d\xi_2 + \left(M_{12}(\xi_2)\delta u_{3\,\xi_1 = \text{constant}} \right)_{a_2}^{b_2} \tag{67}$$

equations (66) are expressed as

$$\int_{\partial A} \delta \mathscr{W}_E^B \, ds = \int_{a_2}^{b_2} \left[\delta \widehat{\mathscr{W}}_{E1}^B A_2 \right]_{\xi_1 = \text{constant}} d\xi_2 - \left(M_{12}(\xi_2) \delta u_{3_{\xi_1 = \text{constant}}} \right)_{a_2}^{b_2} \tag{68a}$$

with

$$\delta \widehat{\mathscr{W}}_{E1}^B = \left[N_1(\xi_2) + \frac{c_3 M_1(\xi_2)}{R_1} \right] \delta u_1 + \left[S_1(\xi_2) + \frac{c_3 M_{12}(\xi_2)}{R_2} \right] \delta u_2 + \left[Q_1(\xi_2) + \frac{dM_{12}(\xi_2)}{d\xi_2} \right] \delta u_3 - \frac{M_1(\xi_2)}{A_1} \frac{\partial \delta u_3}{\partial \xi_1} \tag{68b}$$

The matrix form of equation (68b) is given by

$$\delta \widehat{\mathscr{W}}_{E1}^B = \left\{ \overline{\boldsymbol{\mathscr{N}}}_{\mathbf{1}} \right\}^T \left\{ \boldsymbol{\delta u} \right\} - \left\{ \overline{\boldsymbol{\mathscr{M}}}_{\mathbf{1}} \right\}^T \left[\mathbf{k}_{\mathbf{11}} \right] \frac{1}{A_1} \frac{\partial \left\{ \boldsymbol{\delta u} \right\}}{\partial \xi_1} \tag{69}$$

with

$$\left\{ \overline{\boldsymbol{\mathscr{N}}}_{\mathbf{1}} \right\} \equiv \begin{Bmatrix} N_1 + \dfrac{c_3 M_1}{R_1} \\ S_1 + \dfrac{c_3 M_{12}}{R_2} \\ Q_1 + \dfrac{dM_{12}}{d\xi_2} \end{Bmatrix} \tag{70a}$$

$$\left\{ \overline{\boldsymbol{\mathscr{M}}}_{\mathbf{1}} \right\} \equiv \begin{Bmatrix} M_1 \\ 0 \\ 0 \end{Bmatrix} \tag{70b}$$

In addition,

$$\left(M_{12}(\xi_2) \delta u_{3_{\xi_1 = \text{constant}}} \right)_{a_2}^{b_2} = \left(\left[\left\{ \overline{\boldsymbol{\mathscr{M}}}_{\mathbf{1}} \right\}^T \left[\mathbf{k}_{\mathbf{12}} \right] \left\{ \boldsymbol{\delta u} \right\} \right]_{\xi_1 = \text{constant}} \right)_{a_2}^{b_2} \tag{71}$$

Thus, enforcing the boundary integral term in equation (49) for an edge given by ξ_1 = constant by using equations (64b), (65), and (67) and applying the localization lemma of the Calculus of Variations yields

$$\delta\widehat{\mathcal{W}}_{I1}^{B} - \delta\widehat{\mathcal{W}}_{E1}^{B} = \left[\{\mathcal{N}\}^{T}[\mathbf{d}_1] + \{\mathcal{M}\}^{T}[\mathbf{k}_1] + \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\{\mathcal{M}\}^{T}[\mathbf{k}_{12}] \right) - \{\overline{\mathcal{N}}_1\}^{T} \right]\{\boldsymbol{\delta}\mathbf{u}\}$$

$$+ \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \xi_1}\left(A_2\{\mathcal{M}\}^{T}[\mathbf{k}_{11}] \right) + \frac{\partial}{\partial \xi_2}\left(A_1\{\mathcal{M}\}^{T}[\mathbf{k}_{12}] \right) \right]\{\boldsymbol{\delta}\mathbf{u}\} \qquad (72a)$$

$$\left(\{\overline{\mathcal{M}}_1\}^{T} - \{\mathcal{M}\}^{T} \right)[\mathbf{k}_{11}]\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial \xi_1} + \{\mathcal{F}_1\}^{T}\{\boldsymbol{\delta\gamma}^{o}\} = 0$$

$$\left(\left[\left(\{\mathcal{M}\}^{T} - \{\overline{\mathcal{M}}_1\}^{T} \right)[\mathbf{k}_{12}]\{\boldsymbol{\delta}\mathbf{u}\} \right]_{\xi_1 = \text{constant}} \right)_{a_2}^{b_2} = 0 \qquad (72b)$$

where $\{\boldsymbol{\delta}\mathbf{u}\}$, $\frac{1}{A_1}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial \xi_1}$, and $\{\boldsymbol{\delta\gamma}^{o}\}$ are arbitrary virtual displacements. The component form of these equations yield the following boundary conditions for the edge given by ξ_1 = constant:

$$\mathcal{N}_{11}\left(1 + c_1 e_{11}^{o}\right) + \mathcal{N}_{12} c_1\left(e_{12}^{o} - \varphi\right) + \mathcal{M}_{11}\frac{c_3}{R_1} = N_1(\xi_2) + M_1(\xi_2)\frac{c_3}{R_1} \quad \text{or} \quad \delta u_1 = 0 \qquad (73a)$$

$$\mathcal{N}_{12} + \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\mathcal{N}_{11}\left(2e_{12}^{o} + \varphi\right) - \mathcal{N}_{22}\varphi + 2\mathcal{N}_{12} e_{22}^{o} \right]$$
$$+ \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_2} - \frac{1}{R_1} \right) = S_1(\xi_2) + \frac{M_{12}(\xi_2)}{R_2} \quad \text{or} \quad \delta u_2 = 0 \qquad (73b)$$

$$\overline{Q}_{13} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial \xi_2} - \left(\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2 \right) = Q_1(\xi_2) + \frac{dM_{12}(\xi_2)}{d\xi_2} \quad \text{or} \quad \delta u_3 = 0 \qquad (73c)$$

$$\mathcal{M}_{11} = M_1(\xi_2) \quad \text{or} \quad \frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1} = 0 \qquad (73d)$$

$$\mathcal{F}_{11} = 0 \quad \text{or} \quad \delta\gamma_{13}^{o} = 0 \qquad (73e)$$

$$\mathcal{F}_{12} = 0 \quad \text{or} \quad \delta\gamma_{23}^{o} = 0 \qquad (73f)$$

and

$$\mathcal{M}_{12} = M_{12}(\xi_2) \quad \text{or} \quad \delta u_3 = 0 \qquad (73g)$$

at the corners given by $\xi_2 = a_2$ and $\xi_2 = b_2$. Examination of equations (20c) reveals that $\mathcal{F}_{11}$ and

$\mathcal{F}_{12}$ appearing in these boundary conditions correspond to forces per unit length associated with through-the-thickness distributions of σ_{11} and σ_{12}, respectively, that supress transverse-shearing deformations of the plate edge face.

On an edge given by ξ_2 = constant; $d\xi_2 = 0$, $(\hat{N} \bullet \hat{a}_1) = 0$, and $(\hat{N} \bullet \hat{a}_2) = 1$. For this case,

$$\int_{\partial A} \delta \mathcal{W}_I^B \, ds = \int_{a_1}^{b_1} \left[\delta \widehat{\mathcal{W}}_{I2}^B A_1 \right]_{\xi_2 = \text{constant}} d\xi_1 + \left(\left\langle \{\mathcal{M}\}^T [\mathbf{k}_{12}] \{\boldsymbol{\delta}\mathbf{u}\} \right\rangle_{\xi_2 = \text{constant}} \right)_{a_1}^{b_1} \tag{74a}$$

where

$$\left(\left\langle \{\mathcal{M}\}^T [\mathbf{k}_{12}] \{\boldsymbol{\delta}\mathbf{u}\} \right\rangle_{\xi_2 = \text{constant}} \right)_{a_1}^{b_1} = \left(\left[\{\mathcal{M}\}^T [\mathbf{k}_{12}] \{\boldsymbol{\delta}\mathbf{u}\} \right]_{\xi_2 = \text{constant}} \right)_{a_1}^{b_1} \tag{74b}$$

Likewise, equation (33a) has the form

$$\int_{\partial A} \delta \mathcal{W}_E^B \, ds = \int_{a_1}^{b_1} \left[\delta \overline{\overline{\mathcal{W}}}_{E2}^B A_1 \right]_{\xi_2 = \text{constant}} d\xi_1 \tag{75a}$$

with

$$\begin{aligned} \delta \overline{\overline{\mathcal{W}}}_{E2}^B = & \left[S_2(\xi_1) + \frac{c_3 M_{21}(\xi_1)}{R_1} \right] \delta u_1 + \left[N_2(\xi_1) + \frac{c_3 M_2(\xi_1)}{R_2} \right] \delta u_2 + Q_2(\xi_1) \delta u_3 \\ & - \frac{M_2(\xi_1)}{A_2} \frac{\partial \delta u_3}{\partial \xi_2} - \frac{M_{21}(\xi_2)}{A_1} \frac{\partial \delta u_3}{\partial \xi_1} \end{aligned} \tag{75b}$$

where equations (53) have been used for the virtual rotations. Noting that

$$\int_{a_1}^{b_1} \left[M_{21}(\xi_1) \frac{\partial \delta u_3}{\partial \xi_1} \right]_{\xi_2 = \text{constant}} d\xi_1 = -\int_{a_1}^{b_1} \left[\frac{dM_{21}(\xi_1)}{d\xi_1} \delta u_3 \right]_{\xi_2 = \text{constant}} d\xi_1 + \left(M_{21}(\xi_1) \delta u_{3_{\xi_2 = \text{constant}}} \right)_{a_1}^{b_1} \tag{76}$$

equations (75) are expressed as

$$\int_{\partial A} \delta \mathcal{W}_E^B \, ds = \int_{a_1}^{b_1} \left[\delta \widehat{\mathcal{W}}_{E2}^B A_1 \right]_{\xi_2 = \text{constant}} d\xi_1 - \left(M_{21}(\xi_1) \delta u_{3_{\xi_2 = \text{constant}}} \right)_{a_1}^{b_1} \tag{77a}$$

with

$$\delta\widehat{\mathcal{W}}_{E2}^{B} = \left[S_2(\xi_1) + \frac{c_3 M_{21}(\xi_1)}{R_1}\right]\delta u_1 + \left[N_2(\xi_1) + \frac{c_3 M_2(\xi_1)}{R_2}\right]\delta u_2 + \left[Q_2(\xi_1) + \frac{dM_{21}(\xi_1)}{d\xi_1}\right]\delta u_3 - \frac{M_2(\xi_1)}{A_2}\frac{\partial \delta u_3}{\partial \xi_2} \tag{77b}$$

The matrix form of equation (77b) is given by

$$\delta\widehat{\mathcal{W}}_{E2}^{B} = \left\{\overline{\boldsymbol{\mathcal{N}}}_2\right\}^{T}\left\{\boldsymbol{\delta u}\right\} - \left\{\overline{\boldsymbol{\mathcal{M}}}_2\right\}^{T}\left[\mathbf{k}_{22}\right]\frac{1}{A_2}\frac{\partial\left\{\boldsymbol{\delta u}\right\}}{\partial \xi_2} \tag{78}$$

with

$$\left\{\overline{\boldsymbol{\mathcal{N}}}_2\right\} \equiv \begin{Bmatrix} S_2 + \dfrac{c_3 M_{21}}{R_1} \\ N_2 + \dfrac{c_3 M_2}{R_2} \\ Q_2 + \dfrac{dM_{21}}{d\xi_1} \end{Bmatrix} \tag{79a}$$

$$\left\{\overline{\boldsymbol{\mathcal{M}}}_2\right\} \equiv \begin{Bmatrix} 0 \\ M_2 \\ 0 \end{Bmatrix} \tag{79b}$$

In addition,

$$\left(M_{21}(\xi_1)\delta u_{3\,\xi_2 = \text{constant}}\right)_{a_1}^{b_1} = \left(\left[\left\{\overline{\boldsymbol{\mathcal{M}}}_2\right\}^{T}\left[\mathbf{k}_{12}\right]\left\{\boldsymbol{\delta u}\right\}\right]_{\xi_2 = \text{constant}}\right)_{a_1}^{b_1} \tag{80}$$

Thus, enforcing the boundary integral term in equation (49) for an edge given by ξ_2 = constant by using equations (64c), (74), and (76) and applying the localization lemma of the Calculus of Variations yields

$$\begin{aligned}\delta\widehat{\mathcal{W}}_{I2}^{B} - \delta\widehat{\mathcal{W}}_{E2}^{B} = {} & \left[\left\{\boldsymbol{\mathcal{N}}\right\}^{T}\left[\mathbf{d}_2\right] + \left\{\boldsymbol{\mathcal{M}}\right\}^{T}\left[\mathbf{k}_2\right] + \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left(\left\{\boldsymbol{\mathcal{M}}\right\}^{T}\left[\mathbf{k}_{12}\right]\right) - \left\{\overline{\boldsymbol{\mathcal{N}}}_2\right\}^{T}\right]\left\{\boldsymbol{\delta u}\right\} \\ & + \frac{1}{A_1 A_2}\left[\frac{\partial}{\partial \xi_1}\left(A_2\left\{\boldsymbol{\mathcal{M}}\right\}^{T}\left[\mathbf{k}_{12}\right]\right) + \frac{\partial}{\partial \xi_2}\left(A_1\left\{\boldsymbol{\mathcal{M}}\right\}^{T}\left[\mathbf{k}_{22}\right]\right)\right]\left\{\boldsymbol{\delta u}\right\} \\ & \left(\left\{\overline{\boldsymbol{\mathcal{M}}}_2\right\}^{T} - \left\{\boldsymbol{\mathcal{M}}\right\}^{T}\right)\left[\mathbf{k}_{22}\right]\frac{1}{A_2}\frac{\partial\left\{\boldsymbol{\delta u}\right\}}{\partial \xi_2} + \left\{\boldsymbol{\mathcal{F}}_2\right\}^{T}\left\{\boldsymbol{\delta\gamma}^{o}\right\}\end{aligned} \tag{81a}$$

$$\left(\left[\left(\{\mathcal{M}\}^{T} - \{\overline{\mathcal{M}}_{2}\}^{T}\right)[\mathbf{k}_{12}]\{\boldsymbol{\delta}\mathbf{u}\}\right]_{\xi_2 = \text{constant}}\right)_{a_1}^{b_1} = 0 \tag{81b}$$

where $\{\boldsymbol{\delta}\mathbf{u}\}$, $\frac{1}{A_2}\frac{\partial\{\boldsymbol{\delta}\mathbf{u}\}}{\partial\xi_2}$ and $\{\boldsymbol{\delta\gamma}^{o}\}$ are arbitrary virtual displacements. The component form of these equations yield the following boundary conditions for the edge given by ξ_2 = constant:

$$\begin{aligned}\mathcal{N}_{12} - \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\left(\mathcal{N}_{11} - \mathcal{N}_{22}\right)\varphi + 2\left(\mathcal{N}_{22}e^{o}_{12} + \mathcal{N}_{12}e^{o}_{11}\right)\right] \\ + \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_1} - \frac{1}{R_2}\right) = S_2(\xi_1) + \frac{M_{21}(\xi_1)}{R_1}\end{aligned} \quad \text{or} \quad \delta u_1 = 0 \tag{82a}$$

$$\mathcal{N}_{22}\left(1 + c_1 e^{o}_{22}\right) + \mathcal{N}_{12}c_1\left(e^{o}_{12} + \varphi\right) + \mathcal{M}_{22}\frac{c_3}{R_2} = N_2(\xi_1) + M_2(\xi_1)\frac{c_3}{R_2} \quad \text{or} \quad \delta u_2 = 0 \tag{82b}$$

$$\overline{Q}_{23} + \frac{1}{A_1}\frac{\partial\mathcal{M}_{12}}{\partial\xi_1} - \left(\mathcal{N}_{12}\varphi_1 + \mathcal{N}_{22}\varphi_2\right) = Q_2(\xi_1) + \frac{dM_{21}(\xi_1)}{d\xi_1} \quad \text{or} \quad \delta u_3 = 0 \tag{82c}$$

$$\mathcal{M}_{22} = M_2(\xi_1) \quad \text{or} \quad \frac{1}{A_2}\frac{\partial\delta u_3}{\partial\xi_2} = 0 \tag{82d}$$

$$\mathcal{F}_{21} = 0 \quad \text{or} \quad \delta\gamma^{o}_{13} = 0 \tag{82e}$$

$$\mathcal{F}_{22} = 0 \quad \text{or} \quad \delta\gamma^{o}_{23} = 0 \tag{82f}$$

and

$$\mathcal{M}_{12} = M_{21}(\xi_1) \quad \text{or} \quad \delta u_3 = 0 \tag{82g}$$

at the corners given by $\xi_1 = a_1$ and $\xi_1 = b_1$. Examination of equations (20d) reveals that $\mathcal{F}_{12}$ and $\mathcal{F}_{22}$ appearing in these boundary conditions correspond to forces per unit length associated with through-the-thickness distributions of σ_{12} and σ_{22}, respectively, that supress transverse-shearing deformations of the plate edge face.

Alternate Form of the Boundary Conditions

In the present derivation of the boundary conditions, and those given by Koiter,[6, 9-10] the derivatives $\frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1} = 0$ and $\frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2} = 0$ have been taken as the basic displacement parameters along the edges ξ_1 = constant and ξ_2 = constant, respectively. In contrast, Sanders and Budiansky[5, 8, 11] use the virtual rotations $\delta\varphi_1$ and $\delta\varphi_2$ as the corresponding basic displacement parameters. For these basic displacement parameters, equation (73a) and (73d) are replaced with

$$\mathcal{N}_{11}\left(1 + c_1 e^{\circ}_{11}\right) + \mathcal{N}_{12} c_1\left(e^{\circ}_{12} - \varphi\right) = N_1(\xi_2) \quad \text{or} \quad \delta u_1 = 0 \tag{83a}$$

$$\mathcal{M}_{11} = M_1(\xi_2) \quad \text{or} \quad \delta\varphi_1 = 0 \tag{83b}$$

for the edge ξ_1 = constant. Similarly, equation (82b) and (82d) are replaced with

$$\mathcal{N}_{22}\left(1 + c_1 e^{\circ}_{22}\right) + \mathcal{N}_{12} c_1\left(e^{\circ}_{12} + \varphi\right) = N_2(\xi_1) \quad \text{or} \quad \delta u_2 = 0 \tag{84a}$$

$$\mathcal{M}_{22} = M_2(\xi_1) \quad \text{or} \quad \delta\varphi_2 = 0 \tag{84b}$$

As pointed out by Koiter[9] (see part 3, p.49), these alternate boundary conditions are completely equivalent to those presented herein previously, and are completely acceptable.

Thermoelastic Constitutive Equations for Elastic Shells

Up to this point in the present study, the analysis has a very high fidelity within the presumptions of "small" strains and "moderate" rotations. The constitutive equations are approximate in nature and, as a result, are the best place to introduce approximations in a shell theory. The constitutive equations used in the present study are those for a shell made of one or more layers of linear elastic, specially orthotropic materials that are in a state of plane stress. These equations, referred to the shell (ξ_1, ξ_2, ξ_3) coordinate system are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} - \begin{Bmatrix} \bar{\alpha}_{11} \\ \bar{\alpha}_{22} \\ \bar{\alpha}_{12} \end{Bmatrix} \Theta\left(\xi_1, \xi_2, \xi_3\right) \right\} \tag{85a}$$

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \tag{85b}$$

The $\bar{Q}_{ij}$ terms are the transformed, reduced (plane stress) stiffnesses of classical laminated-shell and laminated-plate theories, and the $\bar{C}_{ij}$ terms are the stiffnesses of a generally orthotropic solid. Both the $\bar{Q}_{ij}$ and the $\bar{C}_{ij}$ terms are generally functions of the through-the-thickness coordinate, ξ_3, for a laminated shell. The $\bar{\alpha}_{ij}$ terms are the corresponding transformed coefficients of thermal expansion, and $\Theta(\xi_1, \xi_2, \xi_3)$ is a function that describes the pointwise change in temperature from a uniform reference state. An in-depth description of these quantities is found in references 226 and 227.

The work-conjugate stress resultants defined by equations (20) are the only stress resultants that appear in the virtual work, equilibrium equations, and boundary conditions. Thus, the shell constitutive equations are obtained by substituting equations (15) into (85), and then substituting the result into equations (20). This process gives the two-dimensional shell constitutive equations by the general form

$$\begin{bmatrix} \{\mathcal{N}\} \\ \{\mathcal{M}\} \\ \{\mathcal{F}_1\} \\ \{\mathcal{F}_2\} \\ \{\mathcal{Q}\} \end{bmatrix} = \begin{bmatrix} [\mathbf{C}_{00}] & [\mathbf{C}_{01}] & [\mathbf{C}_{02}] & [\mathbf{C}_{03}] & [\mathbf{C}_{04}] \\ [\mathbf{C}_{10}] & [\mathbf{C}_{11}] & [\mathbf{C}_{12}] & [\mathbf{C}_{13}] & [\mathbf{C}_{14}] \\ [\mathbf{C}_{20}] & [\mathbf{C}_{21}] & [\mathbf{C}_{22}] & [\mathbf{C}_{23}] & [\mathbf{C}_{24}] \\ [\mathbf{C}_{30}] & [\mathbf{C}_{31}] & [\mathbf{C}_{32}] & [\mathbf{C}_{33}] & [\mathbf{C}_{34}] \\ [\mathbf{C}_{40}] & [\mathbf{C}_{41}] & [\mathbf{C}_{42}] & [\mathbf{C}_{43}] & [\mathbf{C}_{44}] + [\mathbf{C}_{55}] \end{bmatrix} \begin{bmatrix} \{\boldsymbol{\varepsilon}^o\} \\ \{\boldsymbol{\chi}^o\} \\ \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\{\boldsymbol{\gamma}^o\} \\ \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\{\boldsymbol{\gamma}^o\} \\ \{\boldsymbol{\gamma}^o\} \end{bmatrix} - \begin{bmatrix} \{\boldsymbol{\Theta}_0\} \\ \{\boldsymbol{\Theta}_1\} \\ \{\boldsymbol{\Theta}_2\} \\ \{\boldsymbol{\Theta}_3\} \\ \{\boldsymbol{\Theta}_4\} \end{bmatrix} \tag{86}$$

where

$$[\mathbf{C}_{ij}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)} [\mathbf{S}_i]^{\mathbf{T}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} [\mathbf{S}_j] d\xi_3 \quad \text{for} \quad i, j \in \{0, 1, 2, 3, 4\} \tag{87a}$$

$$[\mathbf{C}_{55}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)} [\mathbf{S}_5]^{\mathbf{T}} \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix} [\mathbf{S}_5] d\xi_3 \tag{87b}$$

$$\left\{\boldsymbol{\Theta}_{\mathbf{k}}\right\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbf{S}_{\mathbf{k}}\right]^{\mathbf{T}} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \overline{\alpha}_{11} \\ \overline{\alpha}_{22} \\ \overline{\alpha}_{12} \end{Bmatrix} \Theta\left(\xi_1, \xi_2, \xi_3\right) d\xi_3 \tag{88}$$

Substituting equations (17) into equation (87) and using the shorthand notation $z_1 = 1 + \frac{\xi_3}{R_1}$, $z_2 = 1 + \frac{\xi_3}{R_2}$, and $Z = z_1 + z_2 + \frac{1}{2}\left(z_2 - z_1\right)^2$ yields the following exact expressions

$$\left[\mathbf{C}_{\mathbf{00}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}\overline{Q}_{11} & \overline{Q}_{12} & \frac{Z}{2z_1}\overline{Q}_{16} \\ \overline{Q}_{12} & \frac{z_1}{z_2}\overline{Q}_{22} & \frac{Z}{2z_2}\overline{Q}_{26} \\ \frac{Z}{2z_1}\overline{Q}_{16} & \frac{Z}{2z_2}\overline{Q}_{26} & \frac{Z^2}{4z_1 z_2}\overline{Q}_{66} \end{bmatrix} d\xi_3 \tag{89a}$$

$$\left[\mathbf{C}_{\mathbf{01}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}\overline{Q}_{11} & \overline{Q}_{12} & \frac{z_1 + z_2}{2z_1}\overline{Q}_{16} \\ \overline{Q}_{12} & \frac{z_1}{z_2}\overline{Q}_{22} & \frac{z_1 + z_2}{2z_2}\overline{Q}_{26} \\ \frac{Z}{2z_1}\overline{Q}_{16} & \frac{Z}{2z_2}\overline{Q}_{26} & \frac{z_1 + z_2}{4z_1 z_2} Z\,\overline{Q}_{66} \end{bmatrix} \xi_3 d\xi_3 \tag{89b}$$

$$\left[\mathbf{C}_{\mathbf{02}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}F_1\left(\xi_3\right)\overline{Q}_{11} & \frac{z_2}{z_1}F_2\left(\xi_3\right)\overline{Q}_{16} \\ F_1\left(\xi_3\right)\overline{Q}_{12} & F_2\left(\xi_3\right)\overline{Q}_{26} \\ \frac{Z}{2z_1}F_1\left(\xi_3\right)\overline{Q}_{16} & \frac{Z}{2z_1}F_2\left(\xi_3\right)\overline{Q}_{66} \end{bmatrix} d\xi_3 \tag{89c}$$

$$[\mathbf{C_{03}}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} F_1(\xi_3)\overline{Q}_{16} & F_2(\xi_3)\overline{Q}_{12} \\ \frac{z_1}{z_2}F_1(\xi_3)\overline{Q}_{26} & \frac{z_1}{z_2}F_2(\xi_3)\overline{Q}_{22} \\ \frac{Z}{2z_2}F_1(\xi_3)\overline{Q}_{66} & \frac{Z}{2z_2}F_2(\xi_3)\overline{Q}_{26} \end{bmatrix} d\xi_3 \tag{89d}$$

$$[\mathbf{C_{04}}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \left(\frac{\overline{Q}_{12}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{16}}{\rho_{11}}\right)F_1 & -\left(\frac{\overline{Q}_{16}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{11}}{\rho_{11}}\right)F_2 \\ \left(\frac{z_1}{z_2}\frac{\overline{Q}_{22}}{\rho_{22}} + \frac{\overline{Q}_{26}}{\rho_{11}}\right)F_1 & -\left(\frac{z_1}{z_2}\frac{\overline{Q}_{26}}{\rho_{22}} + \frac{\overline{Q}_{12}}{\rho_{11}}\right)F_2 \\ \frac{Z}{2}\left(\frac{\overline{Q}_{26}}{z_2\rho_{22}} + \frac{\overline{Q}_{66}}{z_1\rho_{11}}\right)F_1 & -\frac{Z}{2}\left(\frac{\overline{Q}_{16}}{z_1\rho_{11}} + \frac{\overline{Q}_{66}}{z_2\rho_{22}}\right)F_2 \end{bmatrix} d\xi_3 \tag{89e}$$

$$[\mathbf{C_{11}}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}\overline{Q}_{11} & \overline{Q}_{12} & \frac{z_1+z_2}{2z_1}\overline{Q}_{16} \\ \overline{Q}_{12} & \frac{z_1}{z_2}\overline{Q}_{22} & \frac{z_1+z_2}{2z_2}\overline{Q}_{26} \\ \frac{z_1+z_2}{2z_1}\overline{Q}_{16} & \frac{z_1+z_2}{2z_2}\overline{Q}_{26} & \frac{(z_1+z_2)^2}{4z_1z_2}\overline{Q}_{66} \end{bmatrix} (\xi_3)^2 d\xi_3 \tag{89f}$$

$$[\mathbf{C_{12}}] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}F_1(\xi_3)\overline{Q}_{11} & \frac{z_2}{z_1}F_2(\xi_3)\overline{Q}_{16} \\ F_1(\xi_3)\overline{Q}_{12} & F_2(\xi_3)\overline{Q}_{26} \\ \frac{z_1+z_2}{2z_1}F_1(\xi_3)\overline{Q}_{16} & \frac{z_1+z_2}{2z_1}F_2(\xi_3)\overline{Q}_{66} \end{bmatrix} \xi_3 d\xi_3 \tag{89g}$$

$$\left[\mathbf{C_{13}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} F_1(\xi_3)\overline{Q}_{16} & F_2(\xi_3)\overline{Q}_{12} \\ \frac{z_1}{z_2}F_1(\xi_3)\overline{Q}_{26} & \frac{z_1}{z_2}F_2(\xi_3)\overline{Q}_{22} \\ \frac{z_1+z_2}{2z_2}F_1(\xi_3)\overline{Q}_{66} & \frac{z_1+z_2}{2z_2}F_2(\xi_3)\overline{Q}_{26} \end{bmatrix} \xi_3 d\xi_3 \tag{89h}$$

$$\left[\mathbf{C_{14}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \left(\frac{\overline{Q}_{12}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{16}}{\rho_{11}}\right)F_1 & -\left(\frac{\overline{Q}_{16}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{11}}{\rho_{11}}\right)F_2 \\ \left(\frac{z_1}{z_2}\frac{\overline{Q}_{22}}{\rho_{22}} + \frac{\overline{Q}_{26}}{\rho_{11}}\right)F_1 & -\left(\frac{z_1}{z_2}\frac{\overline{Q}_{26}}{\rho_{22}} + \frac{\overline{Q}_{12}}{\rho_{11}}\right)F_2 \\ \frac{z_1+z_2}{2z_1z_2}\left(\frac{z_1}{\rho_{22}}\overline{Q}_{26} + \frac{z_2}{\rho_{11}}\overline{Q}_{66}\right)F_1 & -\frac{z_1+z_2}{2z_1z_2}\left(\frac{z_1}{\rho_{22}}\overline{Q}_{66} + \frac{z_2}{\rho_{11}}\overline{Q}_{16}\right)F_2 \end{bmatrix} \xi_3 d\xi_3 \tag{89i}$$

$$\left[\mathbf{C_{22}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} F_1^2(\xi_3)\overline{Q}_{11} & F_1(\xi_3)F_2(\xi_3)\overline{Q}_{16} \\ F_1(\xi_3)F_2(\xi_3)\overline{Q}_{16} & F_2^2(\xi_3)\overline{Q}_{66} \end{bmatrix} \frac{z_2}{z_1} d\xi_3 \tag{89j}$$

$$\left[\mathbf{C_{23}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} F_1^2(\xi_3)\overline{Q}_{16} & F_1(\xi_3)F_2(\xi_3)\overline{Q}_{12} \\ F_1(\xi_3)F_2(\xi_3)\overline{Q}_{66} & F_2^2(\xi_3)\overline{Q}_{26} \end{bmatrix} d\xi_3 \tag{89k}$$

$$\left[\mathbf{C_{24}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \left(\frac{\overline{Q}_{12}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{16}}{\rho_{11}}\right)F_1^2 & -\left(\frac{\overline{Q}_{16}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{11}}{\rho_{11}}\right)F_1F_2 \\ \left(\frac{\overline{Q}_{26}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{66}}{\rho_{11}}\right)F_1F_2 & -\left(\frac{\overline{Q}_{66}}{\rho_{22}} + \frac{z_2}{z_1}\frac{\overline{Q}_{16}}{\rho_{11}}\right)F_2^2 \end{bmatrix} d\xi_3 \tag{89l}$$

$$\left[\mathbf{C_{33}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} F_1^2(\xi_3)\bar{Q}_{66} & F_1(\xi_3)F_2(\xi_3)\bar{Q}_{26} \\ F_1(\xi_3)F_2(\xi_3)\bar{Q}_{26} & F_2^2(\xi_3)\bar{Q}_{22} \end{bmatrix} \frac{z_1}{z_2}\, d\xi_3 \tag{89m}$$

$$\left[\mathbf{C_{34}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \left(\frac{\bar{Q}_{66}}{\rho_{11}} + \frac{z_1}{z_2}\frac{\bar{Q}_{26}}{\rho_{22}}\right)F_1^2 & -\left(\frac{\bar{Q}_{16}}{\rho_{11}} + \frac{z_1}{z_2}\frac{\bar{Q}_{66}}{\rho_{22}}\right)F_1F_2 \\ \left(\frac{\bar{Q}_{26}}{\rho_{11}} + \frac{z_1}{z_2}\frac{\bar{Q}_{22}}{\rho_{22}}\right)F_1F_2 & -\left(\frac{\bar{Q}_{12}}{\rho_{11}} + \frac{z_1}{z_2}\frac{\bar{Q}_{26}}{\rho_{22}}\right)F_2^2 \end{bmatrix} d\xi_3 \tag{89n}$$

$$\left[\mathbf{C_{44}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \left(\frac{z_2}{z_1}\frac{\bar{Q}_{66}}{\rho_{11}^2} + \frac{2\bar{Q}_{26}}{\rho_{11}\rho_{22}} + \frac{z_1}{z_2}\frac{\bar{Q}_{22}}{\rho_{22}^2}\right)F_1^2 & -\left(\frac{z_2}{z_1}\frac{\bar{Q}_{16}}{\rho_{11}^2} + \frac{\bar{Q}_{12}+\bar{Q}_{66}}{\rho_{11}\rho_{22}} + \frac{z_1}{z_2}\frac{\bar{Q}_{26}}{\rho_{22}^2}\right)F_1F_2 \\ -\left(\frac{z_2}{z_1}\frac{\bar{Q}_{16}}{\rho_{11}^2} + \frac{\bar{Q}_{12}+\bar{Q}_{66}}{\rho_{11}\rho_{22}} + \frac{z_1}{z_2}\frac{\bar{Q}_{26}}{\rho_{22}^2}\right)F_1F_2 & \left(\frac{z_2}{z_1}\frac{\bar{Q}_{11}}{\rho_{11}^2} + \frac{2\bar{Q}_{16}}{\rho_{11}\rho_{22}} + \frac{z_1}{z_2}\frac{\bar{Q}_{66}}{\rho_{22}^2}\right)F_2^2 \end{bmatrix} d\xi_3 \tag{89o}$$

$$\left[\mathbf{C_{55}}\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \frac{z_2}{z_1}\left(z_1F_1' - \frac{F_1}{R_1}\right)^2\bar{C}_{55} & \left(z_1F_1' - \frac{F_1}{R_1}\right)\left(z_2F_2' - \frac{F_2}{R_2}\right)\bar{C}_{45} \\ \left(z_1F_1' - \frac{F_1}{R_1}\right)\left(z_2F_2' - \frac{F_2}{R_2}\right)\bar{C}_{45} & \frac{z_1}{z_2}\left(z_2F_2' - \frac{F_2}{R_2}\right)^2\bar{C}_{44} \end{bmatrix} d\xi_3 \tag{89p}$$

and $\left[\mathbf{C_{ji}}\right] = \left[\mathbf{C_{ij}}\right]^{\mathbf{T}}$ for $i, j \in \{0, 1, 2, 3, 4\}$. Next, the thermal stress resultants $\{\mathbf{\Theta_k}\}$ are obtained by first expressing the temperature-change field by

$$\Theta(\xi_1, \xi_2, \xi_3) = \hat{\Theta}(\xi_1, \xi_2)G(\xi_3) \tag{90}$$

such that

$$\{\mathbf{\Theta_k}\} = \hat{\Theta}(\xi_1, \xi_2)\{\bar{\mathbf{\Theta}}_\mathbf{k}\} \tag{91}$$

with

$$\left\{\overline{\boldsymbol{\Theta}}_{\mathbf{k}}\right\} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\mathbf{S}_{\mathbf{k}}\right]^{\mathbf{T}} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \overline{\alpha}_{11} \\ \overline{\alpha}_{22} \\ \overline{\alpha}_{12} \end{Bmatrix} G(\xi_3) d\xi_3 \qquad k \in \{0, 1, 2, 3, 4\} \quad (92)$$

Specific expressions for $\left\{\overline{\boldsymbol{\Theta}}_{\mathbf{k}}\right\}$ are obtained by substituting equations (17) into equation (92).

The constitutive equations given by equations (86), (89), and (92) include the effects of transverse-shearing deformations in a very general manner. Specifically, the constitutive equations of a given transverse-shear-deformation theory depend on the choices for the functions $F_1(\xi_3)$ and $F_2(\xi_3)$, which are required to satisfy $F_1(0) = F_2(0) = 0$ and $F_1'(0) = F_2'(0) = 1$. With these functions specified, expressions for the transverse shearing stresses are then obtained by substituting equations (4e) and (4f) into equation (85b) to obtain

$$\sigma_{13} = \frac{F_1'(\xi_3)\left(1 + \frac{\xi_3}{R_1}\right) - \frac{F_1(\xi_3)}{R_1}}{1 + \frac{\xi_3}{R_1}} \overline{C}_{55}\gamma^{o}_{13} + \frac{F_2'(\xi_3)\left(1 + \frac{\xi_3}{R_2}\right) - \frac{F_2(\xi_3)}{R_2}}{1 + \frac{\xi_3}{R_2}} \overline{C}_{45}\gamma^{o}_{23} \qquad (93a)$$

$$\sigma_{23} = \frac{F_1'(\xi_3)\left(1 + \frac{\xi_3}{R_1}\right) - \frac{F_1(\xi_3)}{R_1}}{1 + \frac{\xi_3}{R_1}} \overline{C}_{45}\gamma^{o}_{13} + \frac{F_2'(\xi_3)\left(1 + \frac{\xi_3}{R_2}\right) - \frac{F_2(\xi_3)}{R_2}}{1 + \frac{\xi_3}{R_2}} \overline{C}_{44}\gamma^{o}_{23} \qquad (93b)$$

In refined transverse-shear-deformation theories, it is desireable, but not always necessary, to specify $F_1(\xi_3)$ and $F_2(\xi_3)$ such that $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$. For example, two similar choices for the pair $F_1(\xi_3)$ and $F_2(\xi_3)$ will yield different stress predictions yet their stiffnesses, obtained from equations (89), will yield nearly identical predictions of overall buckling and vibration responses. In contrast, for accurate stress analyses, one might expect $F_1(\xi_3)$ and $F_2(\xi_3)$ to account for the inhomogeneity found in a general laminated-composite wall construction. Examples of various choices for $F_1(\xi_3)$ and $F_2(\xi_3)$ that have been used in the analysis of plates and shells are found in references 193-195. Some examples are discussed subsequently.

Constitutive equations that correspond to a first-order transverse-shear-deformation theory are obtained by specifying

$$F_1(\xi_3) = F_2(\xi_3) = \xi_3 \qquad (94a)$$

For this fundamental, first-approximation case

$$\sigma_{13} = \frac{\bar{C}_{55}\gamma^{\circ}_{13}}{1 + \frac{\xi_3}{R_1}} + \frac{\bar{C}_{45}\gamma^{\circ}_{23}}{1 + \frac{\xi_3}{R_2}} \tag{94b}$$

$$\sigma_{23} = \frac{\bar{C}_{45}\gamma^{\circ}_{13}}{1 + \frac{\xi_3}{R_1}} + \frac{\bar{C}_{44}\gamma^{\circ}_{23}}{1 + \frac{\xi_3}{R_2}} \tag{94c}$$

Thus, the functions specified by equation (94a) lead to expressions for the transverse-shearing stresses that do not satisfy the traction-free boundary conditions $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$. Moreover, these choices for the distibution of the transverse-shearing stresses and strains, given by equation (94a), do not reflect the inhomogenous through-the-thickness nature of laminated-composite and sandwich shells.

A more robust transverse-shear-deformation theory is obtained by specifying

$$F_1(\xi_3) = F_2(\xi_3) = \xi_3 - \frac{4}{3h^2}(\xi_3)^3 \tag{95a}$$

Upon substituting these functions into equations (3) and neglecting the nonlinearities, it is seen that the displacements U_1 and U_2 are cubic functions of ξ_3 and that U_3 has no dependence on ξ_3 at all. As a result of this character, a transverse-shear-deformation theory based on equation (95a) is referred to herein as a {3, 0} shear-deformation theory. For this case

$$\sigma_{13} = \frac{1 - \left[1 + \frac{2}{3}\frac{\xi_3}{R_1}\right]\left(\frac{2\xi_3}{h}\right)^2}{1 + \frac{\xi_3}{R_1}}\bar{C}_{55}\gamma^{\circ}_{13} + \frac{1 - \left[1 + \frac{2}{3}\frac{\xi_3}{R_2}\right]\left(\frac{2\xi_3}{h}\right)^2}{1 + \frac{\xi_3}{R_2}}\bar{C}_{45}\gamma^{\circ}_{23} \tag{95b}$$

$$\sigma_{23} = \frac{1 - \left[1 + \frac{2}{3}\frac{\xi_3}{R_1}\right]\left(\frac{2\xi_3}{h}\right)^2}{1 + \frac{\xi_3}{R_1}}\bar{C}_{45}\gamma^{\circ}_{13} + \frac{1 - \left[1 + \frac{2}{3}\frac{\xi_3}{R_2}\right]\left(\frac{2\xi_3}{h}\right)^2}{1 + \frac{\xi_3}{R_2}}\bar{C}_{44}\gamma^{\circ}_{23} \tag{95c}$$

The functions specified by equation (95a) also lead to expressions for the transverse-shearing stresses that do not satisfy the traction-free boundary conditions $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$ unless h/R_1 and h/R_2 are negligible. If $\mathcal{h}/\mathcal{R}$ denotes that maximum shell thickness divided by the minimum principal radius of curvature, then

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}_{\xi_3 = \pm\frac{h}{2}} = -\frac{\pm\frac{\mathcal{h}}{3\mathcal{R}}}{1 \pm \frac{\mathcal{h}}{2\mathcal{R}}}\begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix}_{\xi_3 = \pm\frac{h}{2}}\begin{Bmatrix} \gamma^{\circ}_{13} \\ \gamma^{\circ}_{23} \end{Bmatrix} \tag{96}$$

For moderately thick shells with $h/\mathcal{R} = 0.1$, the coefficient in equation (96) involving $h/\mathcal{R}$ is equal to 0.035. This result suggests that the error in the traction-free boundary conditions may be within the error of the constitutive equations for the practical range of values $h/\mathcal{R} \leq 0.1$. However, these choices for the distibution of the transverse-shearing stresses and strains, given by equation (95a), also do not reflect the inhomogenous nature of laminated-composite and sandwich shells.

Expressions for $F_1(\xi_3)$ and $F_2(\xi_3)$ that satisfy $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$ are obtained by assuming cubic polynomials for $F_1(\xi_3)$ and $F_2(\xi_3)$, with a total of eight unknown constants, and then enforcing $F_1(0) = F_2(0) = 0$ and $F_1'(0) = F_2'(0) = 1$ in addition to the four traction boundary conditions. This approach yields

$$F_1(\xi_3) = \xi_3 \left[\frac{1 + \left(\frac{2\xi_3}{h}\right)\left(\frac{h}{6R_1}\right) - \frac{1}{3}\left(\frac{2\xi_3}{h}\right)^2 - \frac{1}{12}\left(\frac{h}{R_1}\right)^2}{1 - \frac{1}{12}\left(\frac{h}{R_1}\right)^2} \right] \tag{97a}$$

$$F_2(\xi_3) = \xi_3 \left[\frac{1 + \left(\frac{2\xi_3}{h}\right)\left(\frac{h}{6R_2}\right) - \frac{1}{3}\left(\frac{2\xi_3}{h}\right)^2 - \frac{1}{12}\left(\frac{h}{R_2}\right)^2}{1 - \frac{1}{12}\left(\frac{h}{R_2}\right)^2} \right] \tag{97b}$$

However, R_1 and R_2 are generally functions of (ξ_1, ξ_2), which violates the requirement that $F_1 = F_1(\xi_3)$ and $F_2 = F_2(\xi_3)$. If the terms involving h/R_1 and h/R_2 are neglected, then equation (95a) is obtained. A simple choice for $F_1(\xi_3)$ and $F_2(\xi_3)$ that satisfies $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$ and avoids the presence of R_1 and R_2 is given by

$$F_1(\xi_3) = F_2(\xi_3) = \xi_3 - \frac{8}{h^2}(\xi_3)^3 + \frac{16}{h^4}(\xi_3)^5 \tag{98}$$

These functions and their first derivatives vanish at $\xi_3 = \pm h/2$.

A highly refined transverse-shear-deformation theory for laminated-composite and sandwich shells is obtained by specifying $F_1(\xi_3)$ and $F_2(\xi_3)$ based on "zigzag" kinematics. This approach has been presented in reference 228 for laminated-composite and sandwich plates. This approach satisfies the traction-free boundary conditions $\sigma_{13} = \sigma_{23} = 0$ at $\xi_3 = \pm h/2$, and yields functional forms for $F_1(\xi_3)$ and $F_2(\xi_3)$ that account through-the-thickness inhomogeneities. Moreover, for a general inhomogeneous shell wall, $F_1(\xi_3)$ and $F_2(\xi_3)$ are found to be different functions.

Simplified Constitutive Equations for Elastic Shells

The exact forms of equations (87) and (88) are obtained by integrating equations (89) and

(92) exactly, once the through-the-thickness distribution of the elements of the constitutive matrix in equation (85) are known. Generally, this process leads to very complicated functional expressions for the stiffnesses defined by equations (89). However, for laminated-composite materials, the elements of the constitutive matrix in equation (85) are modeled as piecewise-constant functions and the integration of equations (87) and (88) poses no problems.

In the present study, the approximate nature of the constitutive equations is exploited to simplify the shell constitutive equations by expanding the functions of $\frac{\xi_3}{R_1}$ and $\frac{\xi_3}{R_2}$ appearing in equations (89) in power series and then neglecting terms that are third-order and higher in $\frac{\xi_3}{R_1}$ and $\frac{\xi_3}{R_2}$. In addition, it is noted that

$$\frac{z_1}{z_2} \approx 1 + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[\xi_3 - \frac{1}{R_2}(\xi_3)^2\right] \tag{99a}$$

$$\frac{z_2}{z_1} \approx 1 + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left[\xi_3 - \frac{1}{R_1}(\xi_3)^2\right] \tag{99b}$$

$$\frac{Z}{2z_1} \approx 1 + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\xi_3 + \frac{1}{4}\left(\frac{3}{R_1^2} + \frac{1}{R_2^2} - \frac{4}{R_1R_2}\right)(\xi_3)^2 \tag{99c}$$

$$\frac{Z}{2z_2} \approx 1 + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\xi_3 + \frac{1}{4}\left(\frac{1}{R_1^2} + \frac{3}{R_2^2} - \frac{4}{R_1R_2}\right)(\xi_3)^2 \tag{99d}$$

$$\frac{Z^2}{4z_1z_2} \approx 1 + \frac{3}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2(\xi_3)^2 \tag{99e}$$

$$\frac{z_1 + z_2}{2z_1} \approx 1 + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left[\xi_3 - \frac{1}{R_1}(\xi_3)^2\right] \tag{99f}$$

$$\frac{z_1 + z_2}{2z_2} \approx 1 + \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left[\xi_3 - \frac{1}{R_2}(\xi_3)^2\right] \tag{99g}$$

$$\frac{z_1 + z_2}{4z_1z_2} Z \approx 1 + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2(\xi_3)^2 \tag{99h}$$

$$\frac{\left(z_1 + z_2\right)^2}{4z_1z_2} \approx 1 + \frac{1}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 \left(\xi_3\right)^2 \tag{99i}$$

Following this process yields the following constitutive equations for a laminated-composite shell:

$$\begin{aligned}
\left[\mathbf{C_{00}}\right] \approx & \begin{bmatrix} A^0_{11} & A^0_{12} & A^0_{16} \\ A^0_{12} & A^0_{22} & A^0_{26} \\ A^0_{16} & A^0_{26} & A^0_{66} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^1_{11} & 0 & \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^1_{16} \\ 0 & \left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^1_{22} & \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^1_{26} \\ \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^1_{16} & \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^1_{26} & 0 \end{bmatrix} \\
& + \tau \begin{bmatrix} \frac{1}{R_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^2_{11} & 0 & \frac{1}{4}\left(\frac{3}{R_1^2} + \frac{1}{R_2^2} - \frac{4}{R_1R_2}\right)A^2_{16} \\ 0 & \frac{1}{R_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\overline{Q}_{22} & \frac{1}{4}\left(\frac{1}{R_1^2} + \frac{3}{R_2^2} - \frac{4}{R_1R_2}\right)A^2_{26} \\ \frac{1}{4}\left(\frac{3}{R_1^2} + \frac{1}{R_2^2} - \frac{4}{R_1R_2}\right)A^2_{16} & \frac{1}{4}\left(\frac{1}{R_1^2} + \frac{3}{R_2^2} - \frac{4}{R_1R_2}\right)A^2_{26} & \frac{3}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 A^2_{66} \end{bmatrix}
\end{aligned} \tag{100a}$$

$$\left[\mathbf{C_{01}}\right] \approx \begin{bmatrix} A^1_{11} & A^1_{12} & A^1_{16} \\ A^1_{12} & A^1_{22} & A^1_{26} \\ A^1_{16} & A^1_{26} & A^1_{66} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^2_{11} & 0 & \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^2_{16} \\ 0 & \left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^2_{22} & \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^2_{26} \\ \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)A^2_{16} & \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)A^2_{26} & 0 \end{bmatrix}$$

$$
+\,\tau\begin{bmatrix}
\frac{1}{R_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)A^{3}_{11} & 0 & \frac{1}{2R_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)A^{3}_{16} \\
0 & \frac{1}{R_2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)A^{3}_{22} & \frac{1}{2R_2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)A^{3}_{26} \\
\frac{1}{4}\left(\frac{3}{R_1^2}+\frac{1}{R_2^2}-\frac{4}{R_1R_2}\right)A^{3}_{16} & \frac{1}{4}\left(\frac{1}{R_1^2}+\frac{3}{R_2^2}-\frac{4}{R_1R_2}\right)A^{3}_{26} & \frac{1}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)^2 A^{3}_{66}
\end{bmatrix} \qquad \text{(100b)}
$$

$$
\left[\mathbf{C}_{02}\right] \approx \begin{bmatrix} R^{10}_{11} & R^{20}_{16} \\ R^{10}_{12} & R^{20}_{26} \\ R^{10}_{16} & R^{20}_{66} \end{bmatrix} + \left(\frac{1}{R_2}-\frac{1}{R_1}\right)\begin{bmatrix} R^{11}_{11} & R^{21}_{16} \\ 0 & 0 \\ \frac{1}{2}R^{11}_{16} & \frac{1}{2}R^{21}_{66} \end{bmatrix}
$$

$$
+\,\tau\begin{bmatrix}
\frac{1}{R_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)R^{12}_{11} & \frac{1}{R_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)R^{22}_{16} \\
0 & 0 \\
\frac{1}{4}\left(\frac{3}{R_1^2}+\frac{1}{R_2^2}-\frac{4}{R_1R_2}\right)R^{12}_{16} & \frac{1}{4}\left(\frac{3}{R_1^2}+\frac{1}{R_2^2}-\frac{4}{R_1R_2}\right)R^{22}_{66}
\end{bmatrix} \qquad \text{(100c)}
$$

$$
\left[\mathbf{C}_{03}\right] = \begin{bmatrix} R^{10}_{16} & R^{20}_{12} \\ R^{10}_{26} & R^{20}_{22} \\ R^{10}_{66} & R^{20}_{26} \end{bmatrix} + \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\begin{bmatrix} 0 & 0 \\ R^{11}_{26} & R^{21}_{22} \\ \frac{1}{2}R^{11}_{66} & \frac{1}{2}R^{21}_{26} \end{bmatrix}
$$

$$
+\,\tau\begin{bmatrix}
0 & 0 \\
\frac{1}{R_2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)R^{12}_{26} & \frac{1}{R_2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)R^{22}_{22} \\
\frac{1}{4}\left(\frac{1}{R_1^2}+\frac{3}{R_2^2}-\frac{4}{R_1R_2}\right)R^{12}_{66} & \frac{1}{4}\left(\frac{1}{R_1^2}+\frac{3}{R_2^2}-\frac{4}{R_1R_2}\right)R^{22}_{26}
\end{bmatrix} \qquad \text{(100d)}
$$

$$\left[\mathbf{C_{04}}\right] \approx \frac{1}{2\rho_{11}}\begin{bmatrix} 2R_{16}^{10} + 2\left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{16}^{11} & -2R_{11}^{20} + 2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{11}^{21} \\ 2R_{26}^{10} & -2R_{12}^{20} \\ R_{66}^{10} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{66}^{11} & -R_{16}^{20} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{16}^{21} \end{bmatrix}$$

$$+\frac{1}{2\rho_{22}}\begin{bmatrix} 2R_{12}^{10} & -2R_{16}^{20} \\ 2R_{22}^{10} + 2\left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{22}^{11} & -2R_{26}^{20} + 2\left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{26}^{21} \\ R_{26}^{10} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{26}^{11} & -R_{66}^{20} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{66}^{21} \end{bmatrix}$$

$$+\frac{\tau}{4\rho_{11}}\begin{bmatrix} \frac{4}{R_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{16}^{12} & \frac{4}{R_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{11}^{22} \\ 0 & 0 \\ \left(\frac{3}{R_1^2} + \frac{1}{R_2^2} - \frac{4}{R_1R_2}\right)R_{66}^{11} & -\left(\frac{3}{R_1^2} + \frac{1}{R_2^2} - \frac{4}{R_1R_2}\right)R_{16}^{21} \end{bmatrix}$$

$$+\frac{\tau}{4\rho_{22}}\begin{bmatrix} 0 & 0 \\ \frac{4}{R_2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)R_{22}^{12} & \frac{4}{R_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)R_{26}^{22} \\ \left(\frac{1}{R_1^2} + \frac{3}{R_2^2} - \frac{4}{R_1R_2}\right)R_{26}^{11} & -\left(\frac{1}{R_1^2} + \frac{3}{R_2^2} - \frac{4}{R_1R_2}\right)R_{66}^{21} \end{bmatrix} \quad (100e)$$

$$\left[\mathbf{C_{11}}\right] \approx \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{16}^2 \\ A_{12}^2 & A_{22}^2 & A_{26}^2 \\ A_{16}^2 & A_{26}^2 & A_{66}^2 \end{bmatrix} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\begin{bmatrix} A_{11}^3 - \frac{\tau A_{11}^4}{R_1} & 0 & \frac{1}{2}\left(A_{16}^3 - \frac{\tau A_{16}^4}{R_1}\right) \\ 0 & -A_{22}^3 + \frac{\tau A_{22}^4}{R_2} & -\frac{1}{2}\left(A_{26}^3 - \frac{\tau A_{26}^4}{R_2}\right) \\ \frac{1}{2}\left(A_{16}^3 - \frac{\tau A_{16}^4}{R_1}\right) & -\frac{1}{2}\left(A_{26}^3 - \frac{\tau A_{26}^4}{R_2}\right) & \frac{1}{4}\tau\left(\frac{1}{R_2} - \frac{1}{R_1}\right)A_{66}^4 \end{bmatrix} \quad (100f)$$

$$\left[\mathbf{C}_{12}\right] \approx \begin{bmatrix} R_{11}^{11} & R_{16}^{21} \\ R_{12}^{11} & R_{26}^{21} \\ R_{16}^{11} & R_{66}^{21} \end{bmatrix} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right) \begin{bmatrix} R_{11}^{12} - \frac{\tau R_{11}^{13}}{R_1} & R_{16}^{22} - \frac{\tau R_{16}^{23}}{R_1} \\ 0 & 0 \\ R_{16}^{12} - \frac{\tau R_{16}^{13}}{R_1} & R_{66}^{22} - \frac{\tau R_{66}^{23}}{R_1} \end{bmatrix} \tag{100g}$$

$$\left[\mathbf{C}_{13}\right] \approx \begin{bmatrix} R_{16}^{11} & R_{12}^{21} \\ R_{26}^{11} & R_{22}^{21} \\ R_{66}^{11} & R_{26}^{21} \end{bmatrix} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \begin{bmatrix} 0 & 0 \\ R_{26}^{12} - \frac{\tau R_{26}^{13}}{R_2} & R_{22}^{22} - \frac{\tau R_{22}^{23}}{R_2} \\ R_{66}^{12} - \frac{\tau R_{66}^{13}}{R_2} & R_{26}^{22} - \frac{\tau R_{26}^{23}}{R_2} \end{bmatrix} \tag{100h}$$

$$\begin{aligned} \left[\mathbf{C}_{14}\right] = \frac{1}{\rho_{22}} \begin{bmatrix} R_{12}^{11} & -R_{16}^{21} \\ R_{22}^{11} & -R_{26}^{21} \\ R_{26}^{11} & -R_{66}^{21} \end{bmatrix} + \frac{1}{\rho_{22}}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \begin{bmatrix} 0 & 0 \\ R_{22}^{12} - \frac{\tau R_{22}^{13}}{R_2} & -R_{26}^{22} + \frac{\tau R_{26}^{23}}{R_2} \\ \frac{1}{2}\left(R_{26}^{12} - \frac{\tau R_{26}^{13}}{R_2}\right) & -\frac{1}{2}\left(R_{66}^{22} - \frac{\tau R_{66}^{23}}{R_2}\right) \end{bmatrix} \\ + \frac{1}{\rho_{11}} \begin{bmatrix} R_{16}^{11} & -R_{11}^{21} \\ R_{26}^{11} & -R_{12}^{21} \\ R_{66}^{11} & -R_{16}^{21} \end{bmatrix} + \frac{1}{\rho_{11}}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \begin{bmatrix} R_{16}^{11} + \left(R_{16}^{12} - \frac{\tau R_{16}^{13}}{R_1}\right) & -\left(R_{11}^{22} - \frac{\tau R_{11}^{23}}{R_1}\right) \\ 0 & 0 \\ \frac{1}{2}\left(R_{66}^{12} - \frac{\tau R_{66}^{13}}{R_1}\right) & -\frac{1}{2}\left(R_{16}^{22} - \frac{\tau R_{16}^{23}}{R_1}\right) \end{bmatrix} \end{aligned} \tag{100i}$$

$$\left[\mathbf{C}_{22}\right] \approx \begin{bmatrix} Q_{11}^{110} & Q_{16}^{120} \\ Q_{16}^{120} & Q_{66}^{220} \end{bmatrix} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right) \begin{bmatrix} Q_{11}^{111} - \frac{\tau Q_{11}^{112}}{R_1} & Q_{16}^{121} - \frac{\tau Q_{16}^{122}}{R_1} \\ Q_{16}^{121} - \frac{\tau Q_{16}^{122}}{R_1} & Q_{66}^{221} - \frac{\tau Q_{66}^{222}}{R_1} \end{bmatrix} \tag{100j}$$

$$\left[\mathbf{C}_{23}\right] \approx \begin{bmatrix} Q_{16}^{110} & Q_{12}^{120} \\ Q_{66}^{120} & Q_{26}^{220} \end{bmatrix} \tag{100k}$$

$$
\left[\mathbf{C_{24}}\right] \approx \frac{1}{\rho_{22}}\begin{bmatrix} Q_{12}^{110} & -Q_{16}^{120} \\ Q_{26}^{120} & -Q_{66}^{220} \end{bmatrix} + \frac{1}{\rho_{11}}\begin{bmatrix} Q_{16}^{110} & -Q_{11}^{120} \\ Q_{66}^{120} & -Q_{16}^{220} \end{bmatrix}
$$

$$
+\frac{1}{\rho_{11}}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\begin{bmatrix} Q_{16}^{111}-\frac{\tau Q_{16}^{112}}{R_1} & -Q_{11}^{121}+\frac{\tau Q_{11}^{122}}{R_1} \\ Q_{66}^{121}-\frac{\tau Q_{66}^{122}}{R_1} & -Q_{16}^{221}+\frac{\tau Q_{16}^{222}}{R_1} \end{bmatrix} \tag{100l}
$$

$$
\left[\mathbf{C_{33}}\right] \approx \begin{bmatrix} Q_{66}^{110} & Q_{26}^{120} \\ Q_{26}^{120} & Q_{22}^{220} \end{bmatrix} + \left(\frac{1}{R_1}-\frac{1}{R_2}\right)\begin{bmatrix} Q_{66}^{111}-\frac{\tau Q_{66}^{112}}{R_2} & Q_{26}^{121}-\frac{\tau Q_{26}^{122}}{R_2} \\ Q_{26}^{121}-\frac{\tau Q_{26}^{122}}{R_2} & Q_{22}^{221}-\frac{\tau Q_{22}^{222}}{R_2} \end{bmatrix} \tag{100m}
$$

$$
\left[\mathbf{C_{34}}\right] = \frac{1}{\rho_{22}}\begin{bmatrix} Q_{26}^{110} & -Q_{66}^{120} \\ Q_{22}^{120} & -Q_{26}^{220} \end{bmatrix} + \frac{1}{\rho_{11}}\begin{bmatrix} Q_{66}^{110} & -Q_{16}^{120} \\ Q_{26}^{120} & -Q_{12}^{220} \end{bmatrix}
$$

$$
+\frac{1}{\rho_{22}}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\begin{bmatrix} Q_{26}^{111}-\frac{\tau Q_{26}^{112}}{R_2} & -Q_{66}^{121}+\frac{\tau Q_{66}^{122}}{R_2} \\ Q_{22}^{121}-\frac{\tau Q_{22}^{122}}{R_2} & -Q_{26}^{221}+\frac{\tau Q_{26}^{222}}{R_2} \end{bmatrix} \tag{100n}
$$

$$
\left[\mathbf{C_{44}}\right] = \frac{1}{\rho_{11}^2}\begin{bmatrix} Q_{66}^{110} & -Q_{16}^{120} \\ -Q_{16}^{120} & Q_{11}^{220} \end{bmatrix} + \frac{1}{\rho_{11}^2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\begin{bmatrix} Q_{66}^{111}-\frac{\tau Q_{66}^{112}}{R_1} & -Q_{16}^{121}+\frac{\tau Q_{16}^{122}}{R_1} \\ -Q_{16}^{121}+\frac{\tau Q_{16}^{122}}{R_1} & Q_{11}^{221}-\frac{\tau Q_{11}^{222}}{R_1} \end{bmatrix}
$$

$$
+\frac{1}{\rho_{22}^2}\begin{bmatrix} Q_{22}^{110} & -Q_{26}^{120} \\ -Q_{26}^{120} & Q_{66}^{220} \end{bmatrix} + \frac{1}{\rho_{22}^2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\begin{bmatrix} Q_{22}^{111}-\frac{\tau Q_{22}^{112}}{R_2} & -Q_{26}^{121}+\frac{\tau Q_{26}^{122}}{R_2} \\ -Q_{26}^{121}+\frac{\tau Q_{26}^{122}}{R_2} & Q_{66}^{221}-\frac{\tau Q_{66}^{222}}{R_2} \end{bmatrix} \tag{100o}
$$

$$
+\frac{1}{\rho_{11}\rho_{22}}\begin{bmatrix} 2Q_{26}^{110} & -\left(Q_{12}^{120}+Q_{66}^{120}\right) \\ -\left(Q_{12}^{120}+Q_{66}^{120}\right) & 2Q_{16}^{220} \end{bmatrix}
$$

$$
\begin{aligned}
\left[\mathbf{C}_{55}\right] = &\begin{bmatrix} Z_{55}^{110} & Z_{45}^{120} \\ Z_{45}^{120} & Z_{44}^{220} \end{bmatrix} + \frac{1}{R_1}\begin{bmatrix} Z_{55}^{111} - 2Y_{55}^{110} & Z_{45}^{121} - Y_{45}^{210} \\ Z_{45}^{121} - Y_{45}^{210} & Z_{44}^{221} \end{bmatrix} + \frac{1}{R_2}\begin{bmatrix} Z_{55}^{111} & Z_{45}^{121} - Y_{45}^{120} \\ Z_{45}^{121} - Y_{45}^{120} & Z_{44}^{221} - 2Y_{44}^{220} \end{bmatrix} \\
&+ \frac{\tau}{R_1 R_2}\begin{bmatrix} Z_{55}^{112} - 2Y_{55}^{111} & Z_{45}^{122} - Y_{45}^{121} - Y_{45}^{211} + X_{45}^{120} \\ Z_{45}^{122} - Y_{45}^{121} - Y_{45}^{211} + X_{45}^{120} & Z_{44}^{222} - 2Y_{44}^{221} \end{bmatrix} + \frac{\tau}{R_1^2}\begin{bmatrix} X_{55}^{110} & 0 \\ 0 & 0 \end{bmatrix} \\
&+ \frac{\tau}{R_2^2}\begin{bmatrix} 0 & 0 \\ 0 & X_{44}^{220} \end{bmatrix} + \tau\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\begin{bmatrix} \frac{1}{R_1^2}\left(X_{55}^{111} - \frac{X_{55}^{112}}{R_1}\right) & 0 \\ 0 & \frac{1}{R_2^2}\left(-X_{44}^{221} + \frac{X_{44}^{222}}{R_2}\right) \end{bmatrix}
\end{aligned}
\qquad (100p)
$$

where

$$
\begin{bmatrix} A_{11}^{k} & A_{12}^{k} & A_{16}^{k} \\ A_{12}^{k} & A_{22}^{k} & A_{26}^{k} \\ A_{16}^{k} & A_{26}^{k} & A_{66}^{k} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} (\xi_3)^k d\xi_3 \qquad (101)
$$

$$
\begin{bmatrix} R_{11}^{jk} & R_{12}^{jk} & R_{16}^{jk} \\ R_{12}^{jk} & R_{22}^{jk} & R_{26}^{jk} \\ R_{16}^{jk} & R_{26}^{jk} & R_{66}^{jk} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} F_j(\xi_3)(\xi_3)^k d\xi_3 \qquad (102)
$$

$$
\begin{bmatrix} Q_{11}^{ijk} & Q_{12}^{ijk} & Q_{16}^{ijk} \\ Q_{12}^{ijk} & Q_{22}^{ijk} & Q_{26}^{ijk} \\ Q_{16}^{ijk} & Q_{26}^{ijk} & Q_{66}^{ijk} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} F_i(\xi_3) F_j(\xi_3)(\xi_3)^k d\xi_3 \qquad (103)
$$

$$
\begin{bmatrix} X_{55}^{ijk} & X_{45}^{ijk} \\ X_{45}^{ijk} & X_{44}^{ijk} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \overline{C}_{55} & \overline{C}_{45} \\ \overline{C}_{45} & \overline{C}_{44} \end{bmatrix} F_i(\xi_3) F_j(\xi_3)(\xi_3)^k d\xi_3 \qquad (104)
$$

$$\begin{bmatrix} Y_{55}^{ijk} & Y_{45}^{ijk} \\ Y_{45}^{ijk} & Y_{44}^{ijk} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix} F_i'(\xi_3) F_j(\xi_3) (\xi_3)^k d\xi_3 \tag{105}$$

$$\begin{bmatrix} Z_{55}^{ijk} & Z_{45}^{ijk} \\ Z_{45}^{ijk} & Z_{44}^{ijk} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix} F_i'(\xi_3) F_j'(\xi_3) (\xi_3)^k d\xi_3 \tag{106}$$

In these equations, $\tau = 1$ gives the constitutive equations that are second order in $\frac{\xi_3}{R_1}$ and $\frac{\xi_3}{R_2}$ that are generalizations of those put forth by Flügge[171] for isotropic shells. Setting $\tau = 0$ gives constitutive equations that are first order. It is also important to note that the matrices obtained from equation (101) for $k = 0$, 1, and 2 correspond to the [A], [B], and [D] matrices, respectively, of classical laminated-plate theory (see reference 226). Similarly, the thermal parts of the constitutive equations are given by

$$\left\{\bar{\boldsymbol{\Theta}}_{\mathbf{0}}\right\} = \begin{Bmatrix} h_{11}^{0} \\ h_{22}^{0} \\ h_{12}^{0} \end{Bmatrix} + \begin{Bmatrix} \dfrac{h_{11}^{1}}{R_2} \\ \dfrac{h_{22}^{1}}{R_1} \\ \dfrac{h_{12}^{1}}{2(R_1 + R_2)} \end{Bmatrix} + \frac{\tau}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)^2 \begin{Bmatrix} 0 \\ 0 \\ h_{12}^{2} \end{Bmatrix} \tag{107a}$$

$$\left\{\bar{\boldsymbol{\Theta}}_{\mathbf{1}}\right\} = \begin{Bmatrix} h_{11}^{1} \\ h_{22}^{1} \\ h_{12}^{1} \end{Bmatrix} + \begin{Bmatrix} \dfrac{h_{11}^{2}}{R_2} \\ \dfrac{h_{22}^{2}}{R_1} \\ \dfrac{h_{12}^{2}}{2(R_1 + R_2)} \end{Bmatrix} \tag{107b}$$

$$\left\{\bar{\boldsymbol{\Theta}}_{\mathbf{2}}\right\} = \begin{Bmatrix} g_{11}^{10} \\ g_{12}^{20} \end{Bmatrix} + \frac{1}{R_2}\begin{Bmatrix} g_{11}^{11} \\ g_{12}^{11} \end{Bmatrix} \tag{107c}$$

$$\left\{\bar{\boldsymbol{\Theta}}_{\mathbf{3}}\right\} = \begin{Bmatrix} g_{12}^{10} \\ g_{22}^{20} \end{Bmatrix} + \frac{1}{R_1}\begin{Bmatrix} g_{12}^{11} \\ g_{22}^{21} \end{Bmatrix} \tag{107d}$$

$$\left\{\bar{\boldsymbol{\Theta}}_4\right\} = \frac{1}{\rho_{22}}\begin{Bmatrix} g_{22}^{10} + \frac{g_{22}^{11}}{R_1} \\ -g_{12}^{20} - \frac{g_{12}^{21}}{R_1} \end{Bmatrix} + \frac{1}{\rho_{11}}\begin{Bmatrix} g_{12}^{10} + \frac{g_{12}^{11}}{R_2} \\ -g_{11}^{20} - \frac{g_{11}^{21}}{R_2} \end{Bmatrix} \tag{107e}$$

where

$$\begin{Bmatrix} h_{11}^{k} \\ h_{22}^{k} \\ h_{12}^{k} \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\alpha}_{11} \\ \bar{\alpha}_{22} \\ \bar{\alpha}_{12} \end{Bmatrix} (\xi_3)^k G(\xi_3) d\xi_3 \tag{108}$$

$$\begin{Bmatrix} g_{11}^{jk} \\ g_{22}^{jk} \\ g_{12}^{jk} \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \bar{\alpha}_{11} \\ \bar{\alpha}_{22} \\ \bar{\alpha}_{12} \end{Bmatrix} F_j(\xi_3)(\xi_3)^k G(\xi_3) d\xi_3 \tag{109}$$

By examining equations (94a) and (95a), it is seen that the maximum magnitudes of $\frac{F_1(\xi_3)}{h}$ and $\frac{F_2(\xi_3)}{h}$ are typically greater than zero and less than unity. Using this information, the constitutive equations given by equations (93) are simplified, using binomial expansions of the denominators, to obtain

$$\sigma_{13} \approx \left[F_1'(\xi_3) - \frac{F_1(\xi_3)}{R_1}\left(1 - \tau\frac{\xi_3}{R_1}\right)\right]\bar{C}_{55}\gamma_{13}^{o} + \left[F_2'(\xi_3) - \frac{F_2(\xi_3)}{R_2}\left(1 - \tau\frac{\xi_3}{R_2}\right)\right]\bar{C}_{45}\gamma_{23}^{o} \tag{110a}$$

$$\sigma_{23} \approx \left[F_1'(\xi_3) - \frac{F_1(\xi_3)}{R_1}\left(1 - \tau\frac{\xi_3}{R_1}\right)\right]\bar{C}_{45}\gamma_{13}^{o} + \left[F_2'(\xi_3) - \frac{F_2(\xi_3)}{R_2}\left(1 - \tau\frac{\xi_3}{R_2}\right)\right]\bar{C}_{44}\gamma_{23}^{o} \tag{110b}$$

Special Cases of the Constitutive Equations

In the original shell theories of Sanders, Budiansky, and Koiter; the constitutive equations used are the simplified, first-approximation constitutive equations of classical Love-Kirchhoff linear shell theory. Their constitutive equations follow from neglecting $\frac{\xi_3}{R_1}$ and $\frac{\xi_3}{R_2}$ in expressions for the effective stress resultants given by equations (20) when they are used with equations (85) to determine the constitutive equations. A similar set of constitutive equations are obtained from equations (106) by neglecting all terms involving principal radii of curvature R_1 and R_2. In addition, terms involving the radii of geodesic curvature ρ_{11} and ρ_{22} are neglected. These terms originally entered the constitutive equations through the matrix $[\mathbf{S_4}]$ in equations (87). To see the rationale for neglecting these terms, this matrix is expressed as

$$[\mathbf{S_4}] = \begin{bmatrix} 0 & -\frac{F_2(\xi_3)}{h}\left(\frac{h}{\rho_{11}}\right)\left(1+\frac{\xi_3}{R_2}\right) \\ \frac{F_1(\xi_3)}{h}\left(\frac{h}{\rho_{22}}\right)\left(1+\frac{\xi_3}{R_1}\right) & 0 \\ \frac{F_1(\xi_3)}{h}\left(\frac{h}{\rho_{11}}\right)\left(1+\frac{\xi_3}{R_2}\right) & -\frac{F_2(\xi_3)}{h}\left(\frac{h}{\rho_{22}}\right)\left(1+\frac{\xi_3}{R_1}\right) \end{bmatrix} \tag{111}$$

Next, the fact that the maximum magnitudes of $\frac{F_1(\xi_3)}{h}$ and $\frac{F_2(\xi_3)}{h}$ are typically greater than zero and less than unity is used again. In addition, the radii of geodesic curvature measure the bending of the coordinate curves within the tangent plane at a given point of the reference surface. Typically, ρ_{11} and ρ_{22} are substantially larger than the shell thickness h and, as a result, $\frac{h}{\rho_{11}}$ and $\frac{h}{\rho_{22}}$ have very small relative magnitudes. Therefore, it follows that constitutive matrices based on $[\mathbf{S_4}]$ can be neglected based on the inherent error in the constitutive equations. Based on the same reasoning, equations (110) are approximated as

$$\sigma_{13} \approx F_1'(\xi_3)\bar{C}_{55}\overset{\circ}{\gamma}_{13} + F_2'(\xi_3)\bar{C}_{45}\overset{\circ}{\gamma}_{23} \tag{112a}$$

$$\sigma_{23} \approx F_1'(\xi_3)\bar{C}_{45}\overset{\circ}{\gamma}_{13} + F_2'(\xi_3)\bar{C}_{44}\overset{\circ}{\gamma}_{23} \tag{112b}$$

A detailed derivation of these equations is presented in Appendix B. Additionally, constitutive equations are presented in Appendix B for transverse-shear deformation theories that include a first-order theory, a {3, 0} theory, and a zigzag theory.

Effects of "Small" Initial Geometric Imperfections

The effects of "small" initial geometric imperfections are obtained in the present study by following the approach presented in reference 221. With regard to the strain fields, the effects of "small" initial geometric imperfections appear only in the nonlinear membrane strains given by equations (5). These effects are obtained by replacing the normal displacement $u_3(\xi_1, \xi_2)$ with $u_3(\xi_1, \xi_2) + w^i(\xi_1, \xi_2)$, where $w^i(\xi_1, \xi_2)$ is a known, measured or assumed, distribution of reference-surface deviations along a vector normal to the reference surface at a given point. Then, all terms involving $w^i(\xi_1, \xi_2)$ appearing in a given membrane strain, that correspond to an unloaded state, are subtracted from that given strain. Applying this process to equation (51) yields

$$\varepsilon^o_{11}\left(\xi_1, \xi_2\right) = e^o_{11} + \frac{1}{2}\left(\varphi_1^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e^o_{11}\right)^2 + e^o_{12}\left(e^o_{12} + 2\varphi\right)\right] + c_1 e^o_{11}\frac{w^i}{R_1} - \varphi_1\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} \tag{113a}$$

$$\varepsilon^o_{22}\left(\xi_1, \xi_2\right) = e^o_{22} + \frac{1}{2}\left(\varphi_2^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e^o_{22}\right)^2 + e^o_{12}\left(e^o_{12} - 2\varphi\right)\right] + c_1 e^o_{22}\frac{w^i}{R_2} - \varphi_2\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} \tag{113b}$$

$$\begin{aligned}\gamma^o_{12}\left(\xi_1, \xi_2\right) = 2e^o_{12} + \varphi_1\varphi_2 + c_1\left[e^o_{11}\left(e^o_{12} - \varphi\right) + e^o_{22}\left(e^o_{12} + \varphi\right)\right] \\ + \frac{c_1 w^i}{R_1}\left(e^o_{12} - \varphi\right) + \frac{c_1 w^i}{R_2}\left(e^o_{12} + \varphi\right) - \varphi_1\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} - \varphi_2\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\end{aligned} \tag{113c}$$

By using equations (23), it follows that

$$\delta\left[c_1 e^o_{11}\frac{w^i}{R_1} - \varphi_1\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\right] = c_1\frac{w^i}{R_1}\left[\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \xi_1} - \frac{\delta u_2}{\rho_{11}} + \frac{\delta u_3}{R_1}\right] - \frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\left[\frac{c_3\delta u_1}{R_1} - \frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1}\right] \tag{114a}$$

$$\delta\left[c_1 e^o_{22}\frac{w^i}{R_2} - \varphi_2\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\right] = c_1\frac{w^i}{R_2}\left[\frac{1}{A_2}\frac{\partial \delta u_2}{\partial \xi_2} + \frac{\delta u_1}{\rho_{22}} + \frac{\delta u_3}{R_2}\right] - \frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\left[\frac{c_3\delta u_2}{R_2} - \frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2}\right] \tag{114b}$$

$$\begin{aligned}\delta\left[\frac{c_1 w^i}{R_1}\left(e^o_{12} - \varphi\right) + \frac{c_1 w^i}{R_2}\left(e^o_{12} + \varphi\right) - \varphi_1\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} - \varphi_2\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\right] \\ = \frac{c_1 w^i}{2R_1}\left[\left(1 + c_3\right)\left(\frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} - \frac{\delta u_2}{\rho_{22}}\right) + \left(1 - c_3\right)\left(\frac{}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} + \frac{\delta u_1}{\rho_{11}}\right)\right] \\ + \frac{c_1 w^i}{2R_2}\left[\left(1 + c_3\right)\left(\frac{1}{A_1}\frac{\partial \delta u_2}{\partial \xi_1} + \frac{\delta u_1}{\rho_{11}}\right) + \left(1 - c_3\right)\left(\frac{1}{A_2}\frac{\partial \delta u_1}{\partial \xi_2} - \frac{\delta u_2}{\rho_{22}}\right)\right] \\ - \frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\left[\frac{c_3\delta u_1}{R_1} - \frac{1}{A_1}\frac{\partial \delta u_3}{\partial \xi_1}\right] - \frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\left[\frac{c_3\delta u_2}{R_2} - \frac{1}{A_2}\frac{\partial \delta u_3}{\partial \xi_2}\right]\end{aligned} \tag{114c}$$

Equation (26) is then expressed as

$$\left\{\boldsymbol{\delta\varepsilon}^o\right\} = \left[\mathbf{d}_0 + \mathbf{d}^i_0\right]\left\{\boldsymbol{\delta u}\right\} + \left[\mathbf{d}_1 + \mathbf{d}^i_1\right]\frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left\{\boldsymbol{\delta u}\right\} + \left[\mathbf{d}_2 + \mathbf{d}^i_2\right]\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left\{\boldsymbol{\delta u}\right\} \tag{115}$$

where $[\mathbf{d_0}]$, $[\mathbf{d_1}]$, and $[\mathbf{d_2}]$ are given by equations (55) and where

$$\left[\mathbf{d}_0^{\mathbf{i}}\right] = \begin{bmatrix} -\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\frac{c_3}{R_1} & -\frac{w^i}{R_1}\frac{c_1}{\rho_{11}} & \frac{w^i}{R_1}\frac{c_1}{R_1} \\ \frac{w^i}{R_2}\frac{c_1}{\rho_{22}} & -\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\frac{c_3}{R_2} & \frac{w^i}{R_2}\frac{c_1}{R_2} \\ \frac{w^i}{R_2}\frac{c_1}{\rho_{11}} - \frac{c_3}{R_1}\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} & -\frac{w^i}{R_1}\frac{c_1}{\rho_{22}} - \frac{c_3}{R_2}\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} & 0 \end{bmatrix} \quad (116)$$

$$\left[\mathbf{d}_1^{\mathbf{i}}\right] = \begin{bmatrix} c_1\frac{w^i}{R_1} & 0 & \frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} \\ 0 & 0 & 0 \\ 0 & \frac{c_1 w^i}{R_2} & \frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} \end{bmatrix} \quad (117)$$

$$\left[\mathbf{d}_2^{\mathbf{i}}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{c_1 w^i}{R_2} & \frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} \\ \frac{c_1 w^i}{R_1} & 0 & \frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} \end{bmatrix} \quad (118)$$

From these equations, it follows that

$$\left[\mathbf{d}_0^{\mathbf{i}}\right] - \frac{1}{\rho_{22}}\left[\mathbf{d}_1^{\mathbf{i}}\right] + \frac{1}{\rho_{11}}\left[\mathbf{d}_2^{\mathbf{i}}\right] =$$

$$\begin{bmatrix} -\frac{1}{R_1}\left(\frac{c_3}{A_1}\frac{\partial w^i}{\partial \xi_1} + c_1\frac{w^i}{\rho_{22}}\right) & -\frac{w^i}{R_1}\frac{c_1}{\rho_{11}} & \frac{c_1 w^i}{R_1^2} - \frac{1}{\rho_{22}A_1}\frac{\partial w^i}{\partial \xi_1} \\ \frac{w^i}{R_2}\frac{c_1}{\rho_{22}} & \frac{1}{R_2}\left(\frac{c_1 w^i}{\rho_{11}} - \frac{c_3}{A_2}\frac{\partial w^i}{\partial \xi_2}\right) & \frac{1}{\rho_{11}A_2}\frac{\partial w^i}{\partial \xi_2} + \frac{c_1 w^i}{R_2^2} \\ \frac{c_1 w^i}{\rho_{11}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{c_3}{R_1}\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} & -\frac{c_1 w^i}{\rho_{22}}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{c_3}{R_2}\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} & \frac{1}{\rho_{11}A_1}\frac{\partial w^i}{\partial \xi_1} - \frac{1}{\rho_{22}A_2}\frac{\partial w^i}{\partial \xi_2} \end{bmatrix} \quad (119)$$

As a result of the "small" initial geometric imperfections, three additional terms appear in the counterpart of equation (46a); these terms are

$$\{\boldsymbol{\mathcal{N}}\}^{T}\left[\left[\mathbf{d}_0^{i}\right] - \frac{1}{\rho_{22}}\left[\mathbf{d}_1^{i}\right] + \frac{1}{\rho_{11}}\left[\mathbf{d}_2^{i}\right]\right] - \frac{1}{A_1}\frac{\partial}{\partial \xi_1}\left[\{\boldsymbol{\mathcal{N}}\}^{T}\left[\mathbf{d}_1^{i}\right]\right] - \frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left(\{\boldsymbol{\mathcal{N}}\}^{T}\left[\mathbf{d}_2^{i}\right]\right) \tag{120}$$

Including these additional terms in equations (46) and (50) yields the equilibrium equations

$$\frac{1}{A_1}\frac{\partial \mathcal{N}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{12}}{\partial \xi_2} - \frac{2\mathcal{N}_{12}}{\rho_{11}} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{22}} + \frac{c_3\overline{Q}_{13}}{R_1} + \frac{c_3}{2A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{M}_{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] + \mathcal{P}_1 + q_1 + \mathcal{P}_1^{i} + q_1^{i} = 0 \tag{121a}$$

$$\frac{1}{A_1}\frac{\partial \mathcal{N}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{22}}{\partial \xi_2} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{11}} + \frac{2\mathcal{N}_{12}}{\rho_{22}} + \frac{c_3\overline{Q}_{23}}{R_2} + \frac{c_3}{2A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{M}_{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\right] + \mathcal{P}_2 + q_2 + \mathcal{P}_2^{i} + q_2^{i} = 0 \tag{121b}$$

$$\frac{1}{A_1}\frac{\partial \overline{Q}_{13}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \overline{Q}_{23}}{\partial \xi_2} + \frac{\overline{Q}_{13}}{\rho_{22}} - \frac{\overline{Q}_{23}}{\rho_{11}} - \frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2} + \mathcal{P}_3 + q_3 + \mathcal{P}_3^{i} + q_3^{i} = 0 \tag{121c}$$

where

$$\mathcal{P}_1^{i} = \frac{c_3}{R_1}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right] + \frac{c_1}{A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{N}_{11}\frac{w^{i}}{R_1}\right] + \frac{c_1}{A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{N}_{12}\frac{w^{i}}{R_1}\right] + \frac{c_1 w^{i}}{\rho_{22}}\left[\frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2}\right] - \frac{c_1 w^{i}}{\rho_{11}}\mathcal{N}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{122a}$$

$$\mathcal{P}_2^{i} = \frac{c_3}{R_2}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right] + \frac{c_1}{A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{N}_{12}\frac{w^{i}}{R_2}\right] + \frac{c_1}{A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{N}_{22}\frac{w^{i}}{R_2}\right] + \frac{c_1 w^{i}}{\rho_{11}}\left(\frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2}\right) + \frac{c_1 w^{i}}{\rho_{22}}\mathcal{N}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{122b}$$

$$\mathcal{P}_3^i = \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^i}{\partial\xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^i}{\partial\xi_2}\right] + \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^i}{\partial\xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^i}{\partial\xi_2}\right]$$
$$+ \frac{1}{\rho_{22}}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^i}{\partial\xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^i}{\partial\xi_2}\right] - \frac{1}{\rho_{11}}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^i}{\partial\xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^i}{\partial\xi_2}\right] - c_1\left[\frac{\mathcal{N}_{22}}{R_2}\frac{w^i}{R_2} + \frac{\mathcal{N}_{11}}{R_1}\frac{w^i}{R_1}\right] \quad (122c)$$

and where

$$\begin{Bmatrix} q_1^i \\ q_2^i \\ q_3^i \end{Bmatrix} = \begin{Bmatrix} -q_3^L\dfrac{1}{A_1}\dfrac{\partial w^i}{\partial\xi_1} \\ -q_3^L\dfrac{1}{A_2}\dfrac{\partial w^i}{\partial\xi_2} \\ q_3^L\left(\dfrac{w^i}{R_1} + \dfrac{w^i}{R_2}\right) + \dfrac{\partial q_3^L}{\partial\xi_3}w^i \end{Bmatrix} \quad (122d)$$

The terms in equation (122d) arise from the live pressure load.

Additional terms associated with the initial geometric imperfections also appear in the boundary conditions given by equations (72) and (81). In particular, $\{\boldsymbol{\mathcal{N}}\}^T[\mathbf{d_1}]$ in equation (72a) is replaced with $\{\boldsymbol{\mathcal{N}}\}^T[\mathbf{d_1} + \mathbf{d_1^i}]$, and $\{\boldsymbol{\mathcal{N}}\}^T[\mathbf{d_2}]$ in equation (81a) is replaced with $\{\boldsymbol{\mathcal{N}}\}^T[\mathbf{d_2} + \mathbf{d_2^i}]$. As a result, equations (73a)-(73c) become

$$\mathcal{N}_{11}\left[1 + c_1\left(e_{11}^o + \frac{w^i}{R_1}\right)\right] + \mathcal{N}_{12}c_1\left(e_{12}^o - \varphi\right) + \mathcal{M}_{11}\frac{c_3}{R_1} = N_1(\xi_2) + M_1(\xi_2)\frac{c_3}{R_1} \quad \text{or} \quad \delta u_1 = 0 \quad (123a)$$

$$\mathcal{N}_{12} + \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\mathcal{N}_{11}\left(2e_{12}^o + \varphi\right) - \mathcal{N}_{22}\varphi + 2\mathcal{N}_{12}\left(e_{22}^o + \frac{w^i}{R_2}\right)\right]$$
$$+ \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right) = S_1(\xi_2) + \frac{M_{12}(\xi_2)}{R_2} \quad \text{or} \quad \delta u_2 = 0 \quad (123b)$$

$$\overline{Q}_{13} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial\xi_2} - \left[\mathcal{N}_{11}\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^i}{\partial\xi_1}\right) + \mathcal{N}_{12}\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^i}{\partial\xi_2}\right)\right] = Q_1(\xi_2) + \frac{dM_{12}(\xi_2)}{d\xi_2} \quad \text{or} \quad \delta u_3 = 0 \quad (123c)$$

Likewise, equations (82a)-(82c) become

$$\mathcal{N}_{12} - \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\mathcal{N}_{11}\varphi + \mathcal{N}_{22}\left(2e^{o}_{12} - \varphi\right) + 2\mathcal{N}_{12}\left(e^{o}_{11} + \frac{w^{i}}{R_1}\right)\right] + \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_1} - \frac{1}{R_2}\right) = S_2(\xi_1) + \frac{M_{21}(\xi_1)}{R_1} \quad \text{or} \quad \delta u_1 = 0 \tag{124a}$$

$$\mathcal{N}_{22}\left[1 + c_1\left(e^{o}_{22} + \frac{w^{i}}{R_2}\right)\right] + \mathcal{N}_{12}c_1\left(e^{o}_{12} + \varphi\right) + \mathcal{M}_{22}\frac{c_3}{R_2} = N_2(\xi_1) + M_2(\xi_1)\frac{c_3}{R_2} \quad \text{or} \quad \delta u_2 = 0 \tag{124b}$$

$$\overline{Q}_{23} + \frac{1}{A_1}\frac{\partial \mathcal{M}_{12}}{\partial \xi_1} - \left[\mathcal{N}_{12}\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1}\right) + \mathcal{N}_{22}\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right)\right] = Q_2(\xi_1) + \frac{dM_{21}(\xi_1)}{d\xi_1} \quad \text{or} \quad \delta u_3 = 0 \tag{124c}$$

In addition, the alternate boundary conditions given by equation (83a) becomes

$$\mathcal{N}_{11}\left[1 + c_1\left(e^{o}_{11} + \frac{w^{i}}{R_1}\right)\right] + \mathcal{N}_{12}c_1\left(e^{o}_{12} - \varphi\right) = N_1(\xi_2) \quad \text{or} \quad \delta u_1 = 0 \tag{125}$$

and that given by equation (84a) becomes

$$\mathcal{N}_{22}\left[1 + c_1\left(e^{o}_{22} + \frac{w^{i}}{R_2}\right)\right] + \mathcal{N}_{12}c_1\left(e^{o}_{12} + \varphi\right) = N_2(\xi_1) \quad \text{or} \quad \delta u_2 = 0 \tag{126}$$

Expressions for the displacements U_1, U_2, and U_3 are obtained by replacing $u_3(\xi_1, \xi_2)$ with $u_3(\xi_1, \xi_2) + w^{i}(\xi_1, \xi_2)$ in equations (52) and (3), and then eliminating terms involving $w^{i}(\xi_1, \xi_2)$ in equation (3) that are left over when u_1, u_2, and u_3 are set equal to zero. This process gives

$$U_1(\xi_1, \xi_2, \xi_3) = u_1 + \xi_3\left[\varphi_1 - \varphi\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right)\right] + F_1(\xi_3)\gamma^{o}_{13} \tag{127}$$

$$U_2(\xi_1, \xi_2, \xi_3) = u_2 + \xi_3\left[\varphi_2 + \varphi\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1}\right)\right] + F_2(\xi_3)\gamma^{o}_{23} \tag{128}$$

$$U_3(\xi_1, \xi_2, \xi_3) = u_3 + w^{i} - \xi_3\left[\frac{1}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \varphi_1\frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1} + \varphi_2\frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right] \tag{129}$$

A complete resumé of these fundamental equations is given in Appendix C.

Concluding Remarks

A detailed exposition on a refined nonlinear shell theory that is suitable for nonlinear limit-point buckling analyses of practical laminated-composite aerospace structures has been presented. This shell theory includes the classical nonlinear shell theory attributed to Leonard, Sanders, Koiter, and Budiansky as an explicit proper subset that is obtained directly by neglecting all quantities associated with higher-order effects such as transverse-shearing deformation. This approach has been used in order to leverage the exisiting experience base and to make the theory attractive to industry. The formalism of general tensors has been avoided in order to expose the details needed to fully understand and use the theory in a process leading ultimately to vehicle certification.

The shell theory presented is constructed around a set of strain-displacement relations that are based on "small" strains and "moderate" rotations. No shell-thinness approximations involving the ratio of the maximum thickness to the minimum radius of curvature were used and, as a result, the strain-displacement relations are exact within the presumptions of "small" strains and "moderate" rotations. To faciliate physical insight, these strain-displacement relations have been presented in terms of the linear reference-surface strains, rotations, and changes in curvature and twist that appear in the classical "best" first-approximation linear shell theory attributed to Sanders, Koiter, and Budiansky. The effects of transverse-shearing deformations are included in the strain-displacement relations and kinematic equations, in a very general manner, by using analyst-defined functions to describe the through-the-thickness distributions of transverse-shearing strains. This approach yields a wide range of flexibility to the analyst when confronted with new structural configurations and the need to analyze both global and local response phenomena, and it enables a consistent building-block approach to analysis. The three-dimensional elasticity form of the internal virtual work has been used to obtain the symmetrical effective stress resulants that appear in classical nonlinear shell theory attributed to Leonard, Sanders, Koiter, and Budiansky. The principle of virtual work, including "live" pressure effects, and the surface divergence theorem were used to obtain the nonlinear equilibrium equations and boundary conditions.

A general set of thermoelastic constitutive equations for laminated-composite shells have been derived without using any shell-thinness approximations. Acknowledging the approximate nature of constitutive equations, simplified forms and special cases that may be useful in practice have also been discussed. These special cases span a hierarchy of accuracy that ranges from that of first-order transverse-shear deformation theory, to that of a shear-deformation theory with parabolic through-the-thickness distributions for the transverse-shearing stresses, and to that which includes the use of layerwise zigzag kinematics without introducing additional unknown response functions into the formulation of the boundary-value problem. In addition, the effects of shell-thinness approximations on the constitutive equations have been presented. It is noteworthy that none of the shell-thinness approximations appear outside of the constitutive equations. Furthermore, the effects of "small" initial geometric imperfections have been introduced in a relatively simple manner to obtain a nonlinear shell theory suitable for studying the nonlinear limit-point response. The equations of this theory include tracers that are useful in assessing many approximations that appear in the literature. For convenience, a resumé of the fundamental equations of the theory are given in an appendix. Overall, a hierarchy of nonlinear shell theories have been presented in a detailed and unified manner that is amenable to the prediction of global and local responses and to the development of generic design technology.

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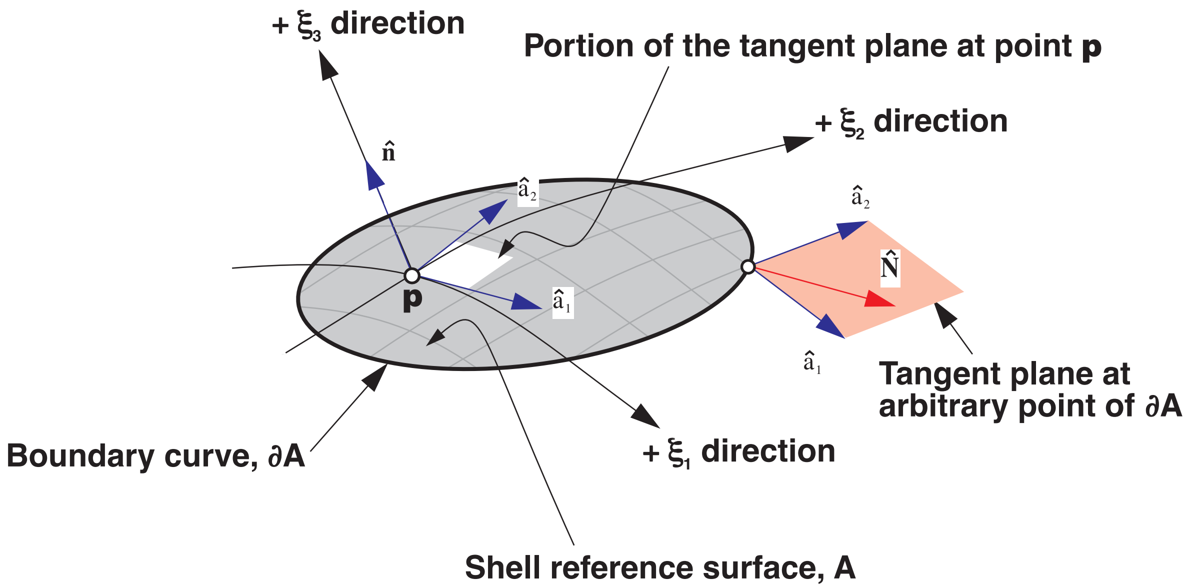

Figure 1. Coordinate system and unit-magnitude base-vector fields for points of undeformed shell.

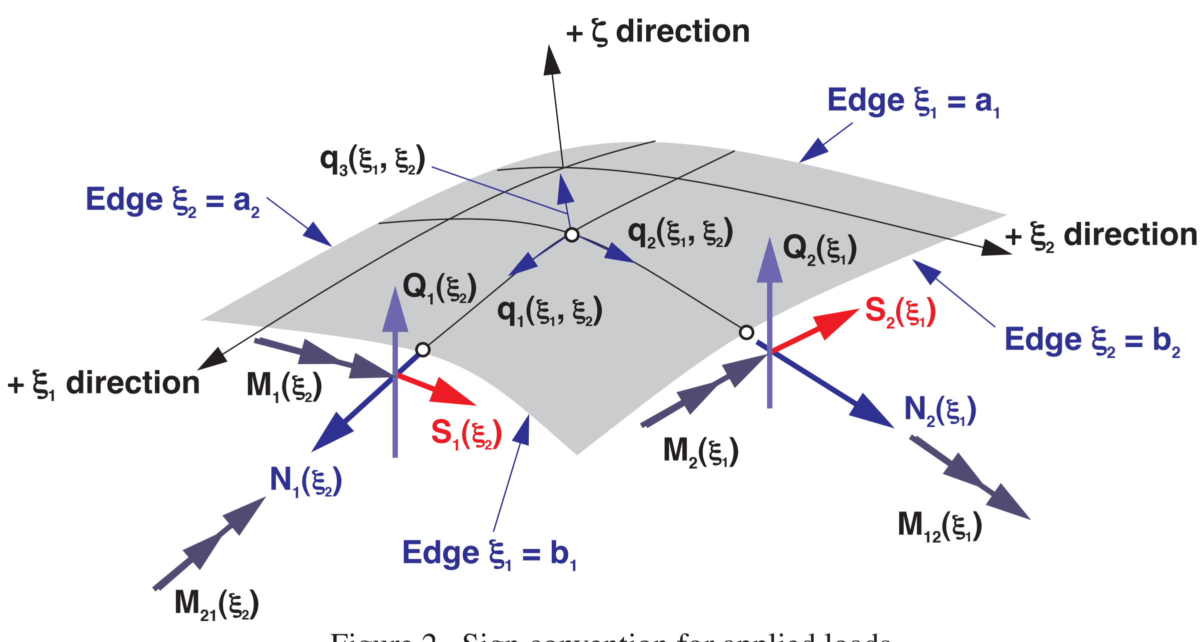

Figure 2. Sign convention for applied loads.

Appendix A - Live Normal Pressure Loads

For a live normal pressure load $\tilde{p}$, the pressure $\tilde{p}$ depends on the deformation of the shell reference surface. Thus,

$$\tilde{p} = q_3\left(\xi_1 + u_1,\ \xi_2 + u_2,\ \xi_3 + u_3\right) \tag{A1}$$

Expanding $\tilde{p}$ in Taylor Series and retaining terms up to first order gives

$$\tilde{p} \approx q_3\left(\xi_1, \xi_2, \xi_3\right) + \left.\frac{\partial q_3}{\partial \xi_1}\right|_{(\xi_1, \xi_2, \xi_3)} u_1\left(\xi_1, \xi_2\right) + \left.\frac{\partial q_3}{\partial \xi_2}\right|_{(\xi_1, \xi_2, \xi_3)} u_2\left(\xi_1, \xi_2\right) + \left.\frac{\partial q_3}{\partial \xi_3}\right|_{(\xi_1, \xi_2, \xi_3)} u_3\left(\xi_1, \xi_2\right) \tag{A2}$$

The differential force due to the live pressure acting on a deformed-shell reference surface is given by

$$d\vec{\tilde{F}} = \tilde{p}\, d\tilde{S}\, \hat{\boldsymbol{n}} \tag{A3}$$

where $\hat{\boldsymbol{n}}\left(\xi_1, \xi_2\right)$ is the unit-magnitude vector field normal to the deformed reference surface and $d\tilde{S}$ is the deformed image of the reference-surface differential area dS. The surface area dS is given in principal-curvature coordinates as

$$dS = A_1 A_2\, d\xi_1 d\xi_2 \tag{A4}$$

For "small" strains and "moderate" rotations, the analysis of reference 221 indicates that

$$d\tilde{S}\hat{\boldsymbol{n}} \approx dS\left[\varphi_1 \hat{a}_1 + \varphi_2 \hat{a}_2 + \left(1 + e^{\circ}_{11} + e^{\circ}_{22}\right)\hat{n}\right] \tag{A5}$$

where $\hat{a}_1\left(\xi_1, \xi_2\right)$, $\hat{a}_2\left(\xi_1, \xi_2\right)$, and $\hat{n}\left(\xi_1, \xi_2\right)$ are the unit-magnitude base-vector fields associated with points of the undeformed reference surface, φ_1 and φ_2 are the rotations defined by equations (8), and e°_{11} and e°_{22} are the linear deformation parameters defined by equations (7). Substituting equations (A2) and (A5) into equation (A3) gives

$$d\vec{\tilde{F}} \approx \left(q_3\varphi_1\hat{a}_1 + q_3\varphi_2\hat{a}_2 + \left[q_3\left(1 + e^{\circ}_{11} + e^{\circ}_{22}\right) + \frac{\partial q_3}{\partial \xi_1} u_1 + \frac{\partial q_3}{\partial \xi_2} u_2 + \frac{\partial q_3}{\partial \xi_3} u_3 \right]\hat{n} \right) dS \tag{A6}$$

where terms involving products of displacements, strains, or rotations are presumed negligible.

Appendix B - Special Cases of the Constitutive Equations

Several special cases of the constitutive equations are presented subsequently in which all terms involving the principal radii of curvature are neglected. Likewise, terms involving the radii of geodesic curvature are also neglected. This approach leads to equations that are consistent with the classical shell theory and the shell theories of Leonard,[7] Sanders,[8] Koiter,[9,10] and Budiansky.[11] Thus, equations (85) reduce to

$$[\mathbf{C}_{00}] \approx \begin{bmatrix} A^0_{11} & A^0_{12} & A^0_{16} \\ A^0_{12} & A^0_{22} & A^0_{26} \\ A^0_{16} & A^0_{26} & A^0_{66} \end{bmatrix} \tag{B1}$$

$$[\mathbf{C}_{01}] \approx \begin{bmatrix} A^1_{11} & A^1_{12} & A^1_{16} \\ A^1_{12} & A^1_{22} & A^1_{26} \\ A^1_{16} & A^1_{26} & A^1_{66} \end{bmatrix} \tag{B2}$$

$$[\mathbf{C}_{02}] \approx \begin{bmatrix} R^{10}_{11} & R^{20}_{16} \\ R^{10}_{12} & R^{20}_{26} \\ R^{10}_{16} & R^{20}_{66} \end{bmatrix} \tag{B3}$$

$$[\mathbf{C}_{03}] = \begin{bmatrix} R^{10}_{16} & R^{20}_{12} \\ R^{10}_{26} & R^{20}_{22} \\ R^{10}_{66} & R^{20}_{26} \end{bmatrix} \tag{B4}$$

$$[\mathbf{C}_{04}] \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{B5}$$

$$[\mathbf{C}_{11}] \approx \begin{bmatrix} A^2_{11} & A^2_{12} & A^2_{16} \\ A^2_{12} & A^2_{22} & A^2_{26} \\ A^2_{16} & A^2_{26} & A^2_{66} \end{bmatrix} \tag{B6}$$

$$[\mathbf{C}_{12}] \approx \begin{bmatrix} R^{11}_{11} & R^{21}_{16} \\ R^{11}_{12} & R^{21}_{26} \\ R^{11}_{16} & R^{21}_{66} \end{bmatrix} \tag{B7}$$

$$[\mathbf{C_{13}}] \approx \begin{bmatrix} R_{16}^{11} & R_{12}^{21} \\ R_{26}^{11} & R_{22}^{21} \\ R_{66}^{11} & R_{26}^{21} \end{bmatrix} \tag{B8}$$

$$[\mathbf{C_{14}}] \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{B9}$$

$$[\mathbf{C_{22}}] \approx \begin{bmatrix} Q_{11}^{110} & Q_{16}^{120} \\ Q_{16}^{120} & Q_{66}^{220} \end{bmatrix} \tag{B10}$$

$$[\mathbf{C_{23}}] \approx \begin{bmatrix} Q_{16}^{110} & Q_{12}^{120} \\ Q_{66}^{120} & Q_{26}^{220} \end{bmatrix} \tag{B11}$$

$$[\mathbf{C_{24}}] \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{B12}$$

$$[\mathbf{C_{33}}] \approx \begin{bmatrix} Q_{66}^{110} & Q_{26}^{120} \\ Q_{26}^{120} & Q_{22}^{220} \end{bmatrix} \tag{B13}$$

$$[\mathbf{C_{34}}] \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{B14}$$

$$[\mathbf{C_{44}}] \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{B15}$$

$$[\mathbf{C_{55}}] = \begin{bmatrix} Z_{55}^{110} & Z_{45}^{120} \\ Z_{45}^{120} & Z_{44}^{220} \end{bmatrix} \tag{B16}$$

In addition, equations (92) become

$$\{\overline{\boldsymbol{\Theta}}_{\mathbf{0}}\} = \begin{Bmatrix} h_{11}^{0} \\ h_{22}^{0} \\ h_{12}^{0} \end{Bmatrix} \tag{B17}$$

$$\left\{\bar{\boldsymbol{\Theta}}_1\right\} = \begin{Bmatrix} h^1_{11} \\ h^1_{22} \\ h^1_{12} \end{Bmatrix} \tag{B18}$$

$$\left\{\bar{\boldsymbol{\Theta}}_2\right\} = \begin{Bmatrix} g^{10}_{11} \\ g^{20}_{12} \end{Bmatrix} \tag{B19}$$

$$\left\{\bar{\boldsymbol{\Theta}}_3\right\} = \begin{Bmatrix} g^{10}_{12} \\ g^{20}_{22} \end{Bmatrix} \tag{B20}$$

$$\left\{\bar{\boldsymbol{\Theta}}_4\right\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{B21}$$

With these simplifications, equation (86) reduces to

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \\ \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = \begin{bmatrix} A^0_{11} & A^0_{12} & A^0_{16} & A^1_{11} & A^1_{12} & A^1_{16} \\ A^0_{12} & A^0_{22} & A^0_{26} & A^1_{12} & A^1_{22} & A^1_{26} \\ A^0_{16} & A^0_{26} & A^0_{66} & A^1_{16} & A^1_{26} & A^1_{66} \\ A^1_{11} & A^1_{12} & A^1_{16} & A^2_{11} & A^2_{12} & A^2_{16} \\ A^1_{12} & A^1_{22} & A^1_{26} & A^2_{12} & A^2_{22} & A^2_{26} \\ A^1_{16} & A^1_{26} & A^1_{66} & A^2_{16} & A^2_{26} & A^2_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_{11} \\ \varepsilon^o_{22} \\ \gamma^o_{12} \\ \chi^o_{11} \\ \chi^o_{22} \\ 2\chi^o_{12} \end{Bmatrix} + \begin{bmatrix} R^{10}_{11} & R^{20}_{16} & R^{10}_{16} & R^{20}_{12} \\ R^{10}_{12} & R^{20}_{26} & R^{10}_{26} & R^{20}_{22} \\ R^{10}_{16} & R^{20}_{66} & R^{10}_{66} & R^{20}_{26} \\ R^{11}_{11} & R^{21}_{16} & R^{11}_{16} & R^{21}_{12} \\ R^{11}_{12} & R^{21}_{26} & R^{11}_{26} & R^{21}_{22} \\ R^{11}_{16} & R^{21}_{66} & R^{11}_{66} & R^{21}_{26} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^o_{13}}{\partial \xi_1} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^o_{23}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^o_{13}}{\partial \xi_2} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^o_{23}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^0_{11} \\ h^0_{22} \\ h^0_{12} \\ h^1_{11} \\ h^1_{22} \\ h^1_{12} \end{Bmatrix} \tag{B22a}$$

$$\begin{Bmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{12} \\ \mathcal{F}_{21} \\ \mathcal{F}_{22} \end{Bmatrix} = \begin{bmatrix} R^{10}_{11} & R^{10}_{12} & R^{10}_{16} & R^{11}_{11} & R^{11}_{12} & R^{11}_{16} \\ R^{20}_{16} & R^{20}_{26} & R^{20}_{66} & R^{21}_{16} & R^{21}_{26} & R^{21}_{66} \\ R^{10}_{16} & R^{10}_{26} & R^{10}_{66} & R^{11}_{16} & R^{11}_{26} & R^{11}_{66} \\ R^{20}_{12} & R^{20}_{22} & R^{20}_{26} & R^{21}_{12} & R^{21}_{22} & R^{21}_{26} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_{11} \\ \varepsilon^o_{22} \\ \gamma^o_{12} \\ \chi^o_{11} \\ \chi^o_{22} \\ 2\chi^o_{12} \end{Bmatrix} + \begin{bmatrix} Q^{110}_{11} & Q^{120}_{16} & Q^{110}_{16} & Q^{120}_{12} \\ Q^{120}_{16} & Q^{220}_{66} & Q^{120}_{66} & Q^{220}_{26} \\ Q^{110}_{16} & Q^{120}_{66} & Q^{110}_{66} & Q^{120}_{26} \\ Q^{120}_{12} & Q^{220}_{26} & Q^{120}_{26} & Q^{220}_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^o_{13}}{\partial \xi_1} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^o_{23}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^o_{13}}{\partial \xi_2} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^o_{23}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} g^{10}_{11} \\ g^{20}_{12} \\ g^{10}_{12} \\ g^{20}_{22} \end{Bmatrix} \tag{B22b}$$

$$\begin{Bmatrix} \mathcal{Q}_{13} \\ \mathcal{Q}_{23} \end{Bmatrix} = \begin{bmatrix} Z^{110}_{55} & Z^{120}_{45} \\ Z^{120}_{45} & Z^{220}_{44} \end{bmatrix} \begin{Bmatrix} \gamma^o_{13} \\ \gamma^o_{23} \end{Bmatrix} \tag{B22c}$$

In addition, the constitutive equations for the transverse-shearing stresses are approximated as

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} F_1'(\xi_3)\overline{C}_{55} & F_2'(\xi_3)\overline{C}_{45} \\ F_1'(\xi_3)\overline{C}_{45} & F_2'(\xi_3)\overline{C}_{44} \end{bmatrix} \begin{Bmatrix} \gamma^{\circ}_{13} \\ \gamma^{\circ}_{23} \end{Bmatrix} \tag{B22d}$$

It is important to note that for $F_1(\xi_3) = F_2(\xi_3)$, equations (B22) reduce to

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \\ \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = \begin{bmatrix} A^{0}_{11} & A^{0}_{12} & A^{0}_{16} & A^{1}_{11} & A^{1}_{12} & A^{1}_{16} \\ A^{0}_{12} & A^{0}_{22} & A^{0}_{26} & A^{1}_{12} & A^{1}_{22} & A^{1}_{26} \\ A^{0}_{16} & A^{0}_{26} & A^{0}_{66} & A^{1}_{16} & A^{1}_{26} & A^{1}_{66} \\ A^{1}_{11} & A^{1}_{12} & A^{1}_{16} & A^{2}_{11} & A^{2}_{12} & A^{2}_{16} \\ A^{1}_{12} & A^{1}_{22} & A^{1}_{26} & A^{2}_{12} & A^{2}_{22} & A^{2}_{26} \\ A^{1}_{16} & A^{1}_{26} & A^{1}_{66} & A^{2}_{16} & A^{2}_{26} & A^{2}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{\circ}_{11} \\ \varepsilon^{\circ}_{22} \\ \gamma^{\circ}_{12} \\ \chi^{\circ}_{11} \\ \chi^{\circ}_{22} \\ 2\chi^{\circ}_{12} \end{Bmatrix} + \begin{bmatrix} R^{10}_{11} & R^{10}_{12} & R^{10}_{16} \\ R^{10}_{12} & R^{10}_{22} & R^{10}_{26} \\ R^{10}_{16} & R^{10}_{26} & R^{10}_{66} \\ R^{11}_{11} & R^{11}_{12} & R^{11}_{16} \\ R^{11}_{12} & R^{11}_{22} & R^{11}_{26} \\ R^{11}_{16} & R^{11}_{26} & R^{11}_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_2} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{0}_{11} \\ h^{0}_{22} \\ h^{0}_{12} \\ h^{1}_{11} \\ h^{1}_{22} \\ h^{1}_{12} \end{Bmatrix} \tag{B23a}$$

$$\begin{Bmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{22} \\ \mathcal{F}_{12} \end{Bmatrix} = \begin{bmatrix} R^{10}_{11} & R^{10}_{12} & R^{10}_{16} & R^{11}_{11} & R^{11}_{12} & R^{11}_{16} \\ R^{10}_{12} & R^{10}_{22} & R^{10}_{26} & R^{11}_{12} & R^{11}_{22} & R^{21}_{26} \\ R^{10}_{16} & R^{10}_{26} & R^{10}_{66} & R^{11}_{16} & R^{11}_{26} & R^{11}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{\circ}_{11} \\ \varepsilon^{\circ}_{22} \\ \gamma^{\circ}_{12} \\ \chi^{\circ}_{11} \\ \chi^{\circ}_{22} \\ 2\chi^{\circ}_{12} \end{Bmatrix} + \begin{bmatrix} Q^{110}_{11} & Q^{110}_{12} & Q^{110}_{16} \\ Q^{110}_{12} & Q^{110}_{22} & Q^{110}_{26} \\ Q^{110}_{16} & Q^{110}_{26} & Q^{110}_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_2} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} g^{10}_{11} \\ g^{10}_{22} \\ g^{10}_{12} \end{Bmatrix} \tag{B23b}$$

$$\begin{Bmatrix} \mathcal{Q}_{13} \\ \mathcal{Q}_{23} \end{Bmatrix} = \begin{bmatrix} Z^{110}_{55} & Z^{110}_{45} \\ Z^{110}_{45} & Z^{110}_{44} \end{bmatrix} \begin{Bmatrix} \gamma^{\circ}_{13} \\ \gamma^{\circ}_{23} \end{Bmatrix} \tag{B23c}$$

where $\mathcal{F}_{12} = \mathcal{F}_{21}$.

First-Order Shear-Deformation Theory

The constitutive equations of a first-order shear-deformation theory are obtained by setting $F_1(\xi_3) = F_2(\xi_3) = \xi_3$ in equations (87)-(94). In particular, equations (87) and (88) become

$$\begin{bmatrix} R^{jk}_{11} & R^{jk}_{12} & R^{jk}_{16} \\ R^{jk}_{12} & R^{jk}_{22} & R^{jk}_{26} \\ R^{jk}_{16} & R^{jk}_{26} & R^{jk}_{66} \end{bmatrix} = \begin{bmatrix} A^{k+1}_{11} & A^{k+1}_{12} & A^{k+1}_{16} \\ A^{k+1}_{12} & A^{k+1}_{22} & A^{k+1}_{26} \\ A^{k+1}_{16} & A^{k+1}_{26} & A^{k+1}_{66} \end{bmatrix} \tag{B24a}$$

$$\begin{bmatrix} Q^{ijk}_{11} & Q^{ijk}_{12} & Q^{ijk}_{16} \\ Q^{ijk}_{12} & Q^{ijk}_{22} & Q^{ijk}_{26} \\ Q^{ijk}_{16} & Q^{ijk}_{26} & Q^{ijk}_{66} \end{bmatrix} = \begin{bmatrix} A^{k+2}_{11} & A^{k+2}_{12} & A^{k+2}_{16} \\ A^{k+2}_{12} & A^{k+2}_{22} & A^{k+2}_{26} \\ A^{k+2}_{16} & A^{k+2}_{26} & A^{k+2}_{66} \end{bmatrix} \tag{B24b}$$

where the right-hand-sides are given by equation (86). Likewise, equation (91) becomes

$$\begin{bmatrix} Z_{55}^{ijk} & Z_{45}^{ijk} \\ Z_{45}^{ijk} & Z_{44}^{ijk} \end{bmatrix} = \begin{bmatrix} A_{55}^{k} & A_{45}^{k} \\ A_{45}^{k} & A_{44}^{k} \end{bmatrix} \tag{B25a}$$

where

$$\begin{bmatrix} A_{55}^{k} & A_{45}^{k} \\ A_{45}^{k} & A_{44}^{k} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \overline{C}_{55} & \overline{C}_{45} \\ \overline{C}_{45} & \overline{C}_{44} \end{bmatrix} (\xi_3)^k d\xi_3 \tag{B25b}$$

Moreover, equation (94) becomes

$$\begin{Bmatrix} g_{11}^{jk} \\ g_{22}^{jk} \\ g_{12}^{jk} \end{Bmatrix} = \begin{Bmatrix} h_{11}^{k+1} \\ h_{22}^{k+1} \\ h_{12}^{k+1} \end{Bmatrix} \tag{B26}$$

where the right-hand-side is given by equation (93). Applying these simplifications to equations (B23) yields

$$\begin{bmatrix} R_{11}^{10} & R_{12}^{10} & R_{16}^{10} \\ R_{12}^{10} & R_{22}^{10} & R_{26}^{10} \\ R_{16}^{10} & R_{26}^{10} & R_{66}^{10} \\ R_{11}^{11} & R_{12}^{11} & R_{16}^{11} \\ R_{12}^{11} & R_{22}^{11} & R_{26}^{11} \\ R_{16}^{11} & R_{26}^{11} & R_{66}^{11} \end{bmatrix} = \begin{bmatrix} A_{11}^{1} & A_{12}^{1} & A_{16}^{1} \\ A_{12}^{1} & A_{22}^{1} & A_{26}^{1} \\ A_{16}^{1} & A_{26}^{1} & A_{66}^{1} \\ A_{11}^{2} & A_{12}^{2} & A_{16}^{2} \\ A_{12}^{2} & A_{22}^{2} & A_{26}^{2} \\ A_{16}^{2} & A_{26}^{2} & A_{66}^{2} \end{bmatrix} \tag{B27}$$

$$\begin{bmatrix} Q_{11}^{110} & Q_{12}^{110} & Q_{16}^{110} \\ Q_{12}^{110} & Q_{22}^{110} & Q_{26}^{110} \\ Q_{16}^{110} & Q_{26}^{110} & Q_{66}^{110} \end{bmatrix} = \begin{bmatrix} A_{11}^{2} & A_{16}^{2} & A_{12}^{2} \\ A_{12}^{2} & A_{26}^{2} & A_{22}^{2} \\ A_{16}^{2} & A_{66}^{2} & A_{26}^{2} \end{bmatrix} \tag{B28}$$

$$\begin{bmatrix} Z_{55}^{110} & Z_{45}^{110} \\ Z_{45}^{110} & Z_{44}^{110} \end{bmatrix} = \begin{bmatrix} A_{55}^{0} & A_{45}^{0} \\ A_{45}^{0} & A_{44}^{0} \end{bmatrix} \tag{B29}$$

$$\begin{Bmatrix} g_{11}^{10} \\ g_{22}^{10} \\ g_{12}^{10} \end{Bmatrix} = \begin{Bmatrix} h_{11}^{1} \\ h_{22}^{1} \\ h_{12}^{1} \end{Bmatrix} \tag{B30}$$

As a result of $\mathcal{F}_{12} = \mathcal{F}_{21}$, Substitution of equations (B27) and (B28) into equations (B23) reveals that $\mathcal{F}_{11} = \mathcal{M}_{11}$, $\mathcal{F}_{22} = \mathcal{M}_{22}$, and $\mathcal{F}_{12} = \mathcal{M}_{12}$. Thus, the constitutive equations are expressed in terms of the nomenclature of classical shell theory as

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \\ \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{o}_{11} \\ \varepsilon^{o}_{22} \\ \gamma^{o}_{12} \\ \chi^{o}_{11} \\ \chi^{o}_{22} \\ 2\chi^{o}_{12} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \\ D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_2} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{0}_{11} \\ h^{0}_{22} \\ h^{0}_{12} \\ h^{1}_{11} \\ h^{1}_{22} \\ h^{1}_{12} \end{Bmatrix} \tag{B31}$$

$$\begin{Bmatrix} \mathcal{Q}_{13} \\ \mathcal{Q}_{23} \end{Bmatrix} = \begin{bmatrix} k_{55}A_{55} & k_{45}A_{45} \\ k_{45}A_{45} & k_{44}A_{44} \end{bmatrix} \begin{Bmatrix} \gamma^{o}_{13} \\ \gamma^{o}_{23} \end{Bmatrix} \tag{B32}$$

where k_{44}, k_{45}, and k_{55} are transverse-shear correction factors that are used to compensate for the fact that the transverse-shearing stresses are uniformly distributed across the shell thickness and, hence, do not vanish on the bounding shell surfaces. Additionally, the two equilibrium equations given by (58d) and (58e) become

$$\mathcal{Q}_{13} = -\frac{\mathcal{M}_{12}}{\rho_{11}} + \frac{\mathcal{M}_{11}}{\rho_{22}} + \frac{1}{A_1}\frac{\partial \mathcal{M}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial \xi_2} \tag{B33}$$

$$\mathcal{Q}_{23} = -\frac{\mathcal{M}_{22}}{\rho_{11}} + \frac{\mathcal{M}_{12}}{\rho_{22}} + \frac{1}{A_1}\frac{\partial \mathcal{M}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{22}}{\partial \xi_2} \tag{B34}$$

Moreover, equations (17e) and (17f) become

$$[\mathbf{S_4}] = \begin{bmatrix} 0 & -\dfrac{\xi_3}{\rho_{11}}\left(1 + \dfrac{\xi_3}{R_2}\right) \\ \dfrac{\xi_3}{\rho_{22}}\left(1 + \dfrac{\xi_3}{R_1}\right) & 0 \\ \dfrac{\xi_3}{\rho_{11}}\left(1 + \dfrac{\xi_3}{R_2}\right) & -\dfrac{\xi_3}{\rho_{22}}\left(1 + \dfrac{\xi_3}{R_1}\right) \end{bmatrix} \tag{B35}$$

$$[\mathbf{S_5}] = \begin{bmatrix} \left(1 + \dfrac{\xi_3}{R_2}\right) & 0 \\ 0 & \left(1 + \dfrac{\xi_3}{R_1}\right) \end{bmatrix} \tag{B36}$$

Applying these matrices to equation (20e) and using equations (13) gives

$$\begin{Bmatrix} \mathcal{Q}_{13} \\ \mathcal{Q}_{23} \end{Bmatrix} = \begin{Bmatrix} Q_{13} + \dfrac{M_{22}}{\rho_{22}} + \dfrac{M_{12}}{\rho_{11}} \\ Q_{23} - \dfrac{M_{11}}{\rho_{11}} - \dfrac{M_{21}}{\rho_{22}} \end{Bmatrix} \tag{B37}$$

For the special case in which the stiffnesses and the thermal coefficients appearing in equations (85) are symmetric through the thickness, the following additional simplifications to the constitutive equations are obtained

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{o}_{11} \\ \varepsilon^{o}_{22} \\ \gamma^{o}_{12} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{0}_{11} \\ h^{0}_{22} \\ h^{0}_{12} \end{Bmatrix} \tag{B38}$$

$$\begin{Bmatrix} \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \chi^{o}_{11} + \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_1} \\ \chi^{o}_{22} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_2} \\ 2\chi^{o}_{12} + \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{1}_{11} \\ h^{1}_{22} \\ h^{1}_{12} \end{Bmatrix} \tag{B39}$$

{3, 0} Shear-Deformation Theory

The constitutive equations of a {3, 0} shear-deformation theory are obtained by setting

$$F_1(\xi_3) = F_2(\xi_3) = \xi_3 - \frac{4}{3h^2}(\xi_3)^3 \tag{B40}$$

in equations (87)-(91) and (94). These functions yield parabolic distributions of transverse-shearing stresses across the shell thickness. Substituting these functions into equation (B22d) reveals that the transverse-shearing stresses vanish at $\xi_3 = \pm h/2$. Next, using equation (B40), equations (96) and (97) become

$$\begin{bmatrix} R^{jk}_{11} & R^{jk}_{12} & R^{jk}_{16} \\ R^{jk}_{12} & R^{jk}_{22} & R^{jk}_{26} \\ R^{jk}_{16} & R^{jk}_{26} & R^{jk}_{66} \end{bmatrix} = \begin{bmatrix} A^{k+1}_{11} & A^{k+1}_{12} & A^{k+1}_{16} \\ A^{k+1}_{12} & A^{k+1}_{22} & A^{k+1}_{26} \\ A^{k+1}_{16} & A^{k+1}_{26} & A^{k+1}_{66} \end{bmatrix} - \frac{4}{3h^2} \begin{bmatrix} A^{k+3}_{11} & A^{k+3}_{12} & A^{k+3}_{16} \\ A^{k+3}_{12} & A^{k+3}_{22} & A^{k+3}_{26} \\ A^{k+3}_{16} & A^{k+3}_{26} & A^{k+3}_{66} \end{bmatrix} \tag{B41}$$

$$\begin{bmatrix} Q_{11}^{ijk} & Q_{12}^{ijk} & Q_{16}^{ijk} \\ Q_{12}^{ijk} & Q_{22}^{ijk} & Q_{26}^{ijk} \\ Q_{16}^{ijk} & Q_{26}^{ijk} & Q_{66}^{ijk} \end{bmatrix} = \begin{bmatrix} A_{11}^{k+2} & A_{12}^{k+2} & A_{16}^{k+2} \\ A_{12}^{k+2} & A_{22}^{k+2} & A_{26}^{k+2} \\ A_{16}^{k+2} & A_{26}^{k+2} & A_{66}^{k+2} \end{bmatrix} - \frac{8}{3h^2} \begin{bmatrix} A_{11}^{k+4} & A_{12}^{k+4} & A_{16}^{k+4} \\ A_{12}^{k+4} & A_{22}^{k+4} & A_{26}^{k+4} \\ A_{16}^{k+4} & A_{26}^{k+4} & A_{66}^{k+4} \end{bmatrix} + \frac{16}{9h^4} \begin{bmatrix} A_{11}^{k+6} & A_{12}^{k+6} & A_{16}^{k+6} \\ A_{12}^{k+6} & A_{22}^{k+6} & A_{26}^{k+6} \\ A_{16}^{k+6} & A_{26}^{k+6} & A_{66}^{k+6} \end{bmatrix} \qquad \text{(B42)}$$

where the right-hand-sides are given by equation (95). Likewise, equation (100) becomes

$$\begin{bmatrix} Z_{55}^{ijk} & Z_{45}^{ijk} \\ Z_{45}^{ijk} & Z_{44}^{ijk} \end{bmatrix} = \begin{bmatrix} A_{55}^{k} & A_{45}^{k} \\ A_{45}^{k} & A_{44}^{k} \end{bmatrix} - \frac{8}{h^2} \begin{bmatrix} A_{55}^{k+2} & A_{45}^{k+2} \\ A_{45}^{k+2} & A_{44}^{k+2} \end{bmatrix} + \frac{16}{h^4} \begin{bmatrix} A_{55}^{k+4} & A_{45}^{k+4} \\ A_{45}^{k+4} & A_{44}^{k+4} \end{bmatrix} \qquad \text{(B43)}$$

and equation (103) becomes

$$\begin{Bmatrix} g_{11}^{jk} \\ g_{22}^{jk} \\ g_{12}^{jk} \end{Bmatrix} = \begin{Bmatrix} h_{11}^{k+1} \\ h_{22}^{k+1} \\ h_{12}^{k+1} \end{Bmatrix} - \frac{4}{3h^2} \begin{Bmatrix} h_{11}^{k+3} \\ h_{22}^{k+3} \\ h_{12}^{k+3} \end{Bmatrix} \qquad \text{(B44)}$$

where the right-hand-side is given by equation (102). Applying these simplifications to equations (B23) gives

$$\begin{bmatrix} R_{11}^{10} & R_{12}^{10} & R_{16}^{10} \\ R_{12}^{10} & R_{22}^{10} & R_{26}^{10} \\ R_{16}^{10} & R_{26}^{10} & R_{66}^{10} \\ R_{11}^{11} & R_{12}^{11} & R_{16}^{11} \\ R_{12}^{11} & R_{22}^{11} & R_{26}^{11} \\ R_{16}^{11} & R_{26}^{11} & R_{66}^{11} \end{bmatrix} = \begin{bmatrix} A_{11}^{1} & A_{12}^{1} & A_{16}^{1} \\ A_{12}^{1} & A_{22}^{1} & A_{26}^{1} \\ A_{16}^{1} & A_{26}^{1} & A_{66}^{1} \\ A_{11}^{2} & A_{12}^{2} & A_{16}^{2} \\ A_{12}^{2} & A_{22}^{2} & A_{26}^{2} \\ A_{16}^{2} & A_{26}^{2} & A_{66}^{2} \end{bmatrix} - \frac{4}{3h^2} \begin{bmatrix} A_{11}^{3} & A_{12}^{3} & A_{16}^{3} \\ A_{12}^{3} & A_{22}^{3} & A_{26}^{3} \\ A_{16}^{3} & A_{26}^{3} & A_{66}^{3} \\ A_{11}^{4} & A_{12}^{4} & A_{16}^{4} \\ A_{12}^{4} & A_{22}^{4} & A_{26}^{4} \\ A_{16}^{4} & A_{26}^{4} & A_{66}^{4} \end{bmatrix} \qquad \text{(B45)}$$

$$\begin{bmatrix} Q_{11}^{110} & Q_{12}^{110} & Q_{16}^{110} \\ Q_{12}^{110} & Q_{22}^{110} & Q_{26}^{110} \\ Q_{16}^{110} & Q_{26}^{110} & Q_{66}^{110} \end{bmatrix} = \begin{bmatrix} A_{11}^{2} & A_{16}^{2} & A_{12}^{2} \\ A_{12}^{2} & A_{26}^{2} & A_{22}^{2} \\ A_{16}^{2} & A_{66}^{2} & A_{26}^{2} \end{bmatrix} - \frac{8}{3h^2} \begin{bmatrix} A_{11}^{4} & A_{16}^{4} & A_{12}^{4} \\ A_{12}^{4} & A_{26}^{4} & A_{22}^{4} \\ A_{16}^{4} & A_{66}^{4} & A_{26}^{4} \end{bmatrix} + \frac{16}{9h^4} \begin{bmatrix} A_{11}^{6} & A_{16}^{6} & A_{12}^{6} \\ A_{12}^{6} & A_{26}^{6} & A_{22}^{6} \\ A_{16}^{6} & A_{66}^{6} & A_{26}^{6} \end{bmatrix} \qquad \text{(B46)}$$

$$\begin{bmatrix} Z_{55}^{110} & Z_{45}^{110} \\ Z_{45}^{110} & Z_{44}^{110} \end{bmatrix} = \begin{bmatrix} A_{55}^{0} & A_{45}^{0} \\ A_{45}^{0} & A_{44}^{0} \end{bmatrix} - \frac{8}{h^2} \begin{bmatrix} A_{55}^{2} & A_{45}^{2} \\ A_{45}^{2} & A_{44}^{2} \end{bmatrix} + \frac{16}{h^4} \begin{bmatrix} A_{55}^{4} & A_{45}^{4} \\ A_{45}^{4} & A_{44}^{4} \end{bmatrix} \qquad \text{(B47)}$$

$$\begin{Bmatrix} g_{11}^{10} \\ g_{22}^{10} \\ g_{12}^{10} \end{Bmatrix} = \begin{Bmatrix} h_{11}^{1} \\ h_{22}^{1} \\ h_{12}^{1} \end{Bmatrix} - \frac{4}{3h^2} \begin{Bmatrix} h_{11}^{3} \\ h_{22}^{3} \\ h_{12}^{3} \end{Bmatrix} \qquad \text{(B48)}$$

For the special case in which the stiffnesess and the thermal coefficients appearing in equations (85) are symmetric through the thickness, the following additional simplifications to the constitutive equations are obtained

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \end{Bmatrix} = \begin{bmatrix} A^{0}_{11} & A^{0}_{12} & A^{0}_{16} \\ A^{0}_{12} & A^{0}_{22} & A^{0}_{26} \\ A^{0}_{16} & A^{0}_{26} & A^{0}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^{o}_{11} \\ \varepsilon^{o}_{22} \\ \gamma^{o}_{12} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{0}_{11} \\ h^{0}_{22} \\ h^{0}_{12} \end{Bmatrix} \tag{B49a}$$

$$\begin{Bmatrix} \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = \begin{bmatrix} A^{2}_{11} & A^{2}_{12} & A^{2}_{16} \\ A^{2}_{12} & A^{2}_{22} & A^{2}_{26} \\ A^{2}_{16} & A^{2}_{26} & A^{2}_{66} \end{bmatrix} \begin{Bmatrix} \chi^{o}_{11} \\ \chi^{o}_{22} \\ 2\chi^{o}_{12} \end{Bmatrix} + \begin{bmatrix} R^{11}_{11} & R^{11}_{12} & R^{11}_{16} \\ R^{11}_{12} & R^{11}_{22} & R^{11}_{26} \\ R^{11}_{16} & R^{11}_{26} & R^{11}_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_2} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} h^{1}_{11} \\ h^{1}_{22} \\ h^{1}_{12} \end{Bmatrix} \tag{B49b}$$

$$\begin{Bmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{22} \\ \mathcal{F}_{12} \end{Bmatrix} = \begin{bmatrix} R^{11}_{11} & R^{11}_{12} & R^{11}_{16} \\ R^{11}_{12} & R^{11}_{22} & R^{21}_{26} \\ R^{11}_{16} & R^{11}_{26} & R^{11}_{66} \end{bmatrix} \begin{Bmatrix} \chi^{o}_{11} \\ \chi^{o}_{22} \\ 2\chi^{o}_{12} \end{Bmatrix} + \begin{bmatrix} Q^{110}_{11} & Q^{110}_{12} & Q^{110}_{16} \\ Q^{110}_{12} & Q^{110}_{22} & Q^{110}_{26} \\ Q^{110}_{16} & Q^{110}_{26} & Q^{110}_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_2} \\ \dfrac{1}{A_1}\dfrac{\partial \gamma^{o}_{23}}{\partial \xi_1} + \dfrac{1}{A_2}\dfrac{\partial \gamma^{o}_{13}}{\partial \xi_2} \end{Bmatrix} - \hat{\Theta} \begin{Bmatrix} g^{10}_{11} \\ g^{10}_{22} \\ g^{10}_{12} \end{Bmatrix} \tag{B49c}$$

Shear-Deformation Theory Based on Zigzag Kinematics

When the transverse-shearing stresses are approximated by equation (B22d), the analysis presented in reference 228 for laminated-composite plates is directly applicable to the present study. In particular, the functions $F_1(\xi_3)$ and $F_2(\xi_3)$ are expressed as

$$F_1(\xi_3) = f_1(\xi_3) + \Psi_1^{(n)}(\xi_3) \tag{B50a}$$

$$F_2(\xi_3) = f_2(\xi_3) + \Psi_2^{(n)}(\xi_3) \tag{B50b}$$

where the ply number $n \in \{1, 2, ..., N\}$ and N is the total number of plies in the laminated wall. The functions $f_1(\xi_3)$ and $f_2(\xi_3)$ are continuous functions with continuous derivatives that are required to satisfy

$$f_1(0) = f_2(0) = 0 \tag{B51a}$$

$$f_1'(-\xi_3) = f_1'(\xi_3) \tag{B51b}$$

$$f_2'\left(-\xi_3\right) = f_2'\left(\xi_3\right) \tag{B51c}$$

$$f_1'\left(\pm\frac{h}{2}\right) = f_2'\left(\pm\frac{h}{2}\right) = 0 \tag{B51d}$$

for all values of $-h/2 \le \xi_3 \le h/2$. The functions $\Psi_1^{(n)}(\xi_3)$ and $\Psi_2^{(n)}(\xi_3)$ are referred to in reference 228 as "zigzag enrichment functions" that account for shell wall inhomogeneity and asymmetry. These functions are given by

$$\Psi_1^{(n)}(\xi_3) = \Phi_1^{(n)}(\xi_3) - \Phi_1^{(M)}(0) + \frac{\xi_3}{2}\left[2 - \frac{G_1}{\overline{C}_{55}^{(1)}}\left(1 - \frac{\xi_3}{h}\right) - \frac{G_1}{\overline{C}_{55}^{(N)}}\left(1 + \frac{\xi_3}{h}\right)\right] \tag{B52a}$$

$$\Psi_2^{(n)}(\xi_3) = \Phi_2^{(n)}(\xi_3) - \Phi_2^{(M)}(0) + \frac{\xi_3}{2}\left[2 - \frac{G_2}{\overline{C}_{44}^{(1)}}\left(1 - \frac{\xi_3}{h}\right) - \frac{G_2}{\overline{C}_{44}^{(N)}}\left(1 + \frac{\xi_3}{h}\right)\right] \tag{B52b}$$

where

$$\Phi_1^{(n)}(\xi_3) = \left(\xi_3 + \frac{h}{2}\right)\left[\frac{G_1}{\overline{C}_{55}^{(n)}} - 1\right] + G_1\sum_{p=1}^{n-1}\left[\frac{1}{\overline{C}_{55}^{(p)}} - \frac{1}{\overline{C}_{55}^{(n)}}\right]h_{(p)} \tag{B53a}$$

$$\Phi_2^{(n)}(\xi_3) = \left(\xi_3 + \frac{h}{2}\right)\left[\frac{G_2}{\overline{C}_{44}^{(n)}} - 1\right] + G_2\sum_{p=1}^{n-1}\left[\frac{1}{\overline{C}_{44}^{(p)}} - \frac{1}{\overline{C}_{44}^{(n)}}\right]h_{(p)} \tag{B53b}$$

$$\Phi_1^{(M)}(0) = \frac{h}{2}\left[\frac{G_1}{\overline{C}_{55}^{(M)}} - 1\right] + G_1\sum_{p=1}^{M-1}\left[\frac{1}{\overline{C}_{55}^{(p)}} - \frac{1}{\overline{C}_{55}^{(M)}}\right]h_{(p)} \tag{B54a}$$

$$\Phi_2^{(M)}(0) = \frac{h}{2}\left[\frac{G_2}{\overline{C}_{44}^{(M)}} - 1\right] + G_2\sum_{p=1}^{M-1}\left[\frac{1}{\overline{C}_{44}^{(p)}} - \frac{1}{\overline{C}_{44}^{(M)}}\right]h_{(p)} \tag{B54b}$$

$$G_1 = \left[\frac{1}{h}\sum_{r=1}^{N}\frac{h_{(r)}}{\overline{C}_{55}^{(r)}}\right]^{-1} \tag{B55a}$$

$$G_2 = \left[\frac{1}{h}\sum_{r=1}^{N}\frac{h_{(r)}}{\overline{C}_{44}^{(r)}}\right]^{-1} \tag{B55b}$$

In equations (B52) and (B54), the superscript M denotes the ply that occupies the shell reference surface, $\xi_3 = 0$. The derivatives of $F_1(\xi_3)$ and $F_2(\xi_3)$ are

$$F_1{}'(\xi_3) = f_1{}'(\xi_3) + \Psi_1^{(n)}{}'(\xi_3) \tag{B56a}$$

$$F_2{}'(\xi_3) = f_2{}'(\xi_3) + \Psi_2^{(n)}{}'(\xi_3) \tag{B56b}$$

where

$$\Psi_1^{(n)}{}'(\xi_3) = \frac{G_1}{\overline{C}_{55}^{(n)}} - \frac{1}{2}\left[\frac{G_1}{\overline{C}_{55}^{(1)}}\left(1 - \frac{2\xi_3}{h}\right) + \frac{G_1}{\overline{C}_{55}^{(N)}}\left(1 + \frac{2\xi_3}{h}\right)\right] \tag{B57a}$$

$$\Psi_2^{(n)}{}'(\xi_3) = \frac{G_2}{\overline{C}_{44}^{(n)}} - \frac{1}{2}\left[\frac{G_2}{\overline{C}_{44}^{(1)}}\left(1 - \frac{2\xi_3}{h}\right) + \frac{G_2}{\overline{C}_{44}^{(N)}}\left(1 + \frac{2\xi_3}{h}\right)\right] \tag{B57b}$$

For a homogeneous shell wall, the zig-zag enrichment terms in equations (B50) and (B56) vanish. Thus, the derivatives of the functions $f_1(\xi_3)$ and $f_2(\xi_3)$ represent distribution of transverse-shearing stresses in a homogeneous shell wall. For the parabolic distribution of transverse shearing stresses commonly found in the technical literature for homogeneous plates,

$$f_1(\xi_3) = f_2(\xi_3) = \xi_3\left[1 - \frac{1}{3}\left(\frac{2\xi_3}{h}\right)^2\right] \tag{B58a}$$

$$f_1{}'(\xi_3) = f_2{}'(\xi_3) = 1 - \left(\frac{2\xi_3}{h}\right)^2 \tag{B58b}$$

The shell-wall stiffnesses and thermal coefficients appearing in equations (B22) are obtained by substituting equations (B50), (B56), and (B58) into equations (101)-(109) and performing the through-the-thickness integrations. The resulting expressions are lengthy and are not presented herein.

Appendix C - Resumé of the Fundamental Equations

The fundamental equations needed to characterize the nonlinear behavior of shells with "small" initial geometric imperfections are presented in this appendix. The parameters c_1, c_2, and c_3 appear in these equations and are used herein to identify other well-known shell theories that are contained within the equations of the present study as special cases. In particular, specifying $F_1(\xi_3) = F_2(\xi_3) = 0$, neglecting the initial geometric imperfections, and setting $c_1 = c_2 = c_3 = 1$ gives the nonlinear shell theories of Budiansky[11] and Koiter.[9,10] Similarly, specifying $c_1 = 0$ and $c_2 = c_3 = 1$ gives Sanders' nonlinear shell theory, and specifying $c_1 = 0$, $c_2 = 0$, and $c_3 = 1$ gives Sanders' nonlinear shell theory with nonlinear rotations about the reference-surface normal neglected. Furthermore, specifying $c_1 = c_2 = c_3 = 0$ gives the Donnell-Mushtari-Vlasov[192] nonlinear shell theory.

Displacements and Strain-Displacement Relations

The fundamental unknown fields in the present study are the reference-surface tangential displacements $u_1(\xi_1, \xi_2)$ and $u_2(\xi_1, \xi_2)$, the normal displacement $u_3(\xi_1, \xi_2)$, and the transverse-shearing strains $\overset{\circ}{\gamma}_{13}(\xi_1, \xi_2)$ and $\overset{\circ}{\gamma}_{23}(\xi_1, \xi_2)$. The corresponding displacements of a material point (ξ_1, ξ_2, ξ_3) are given by

$$U_1(\xi_1, \xi_2, \xi_3) = u_1 + \xi_3\left[\varphi_1 - \varphi\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\right)\right] + F_1(\xi_3)\overset{\circ}{\gamma}_{13} \tag{C1}$$

$$U_2(\xi_1, \xi_2, \xi_3) = u_2 + \xi_3\left[\varphi_2 + \varphi\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1}\right)\right] + F_2(\xi_3)\overset{\circ}{\gamma}_{23} \tag{C2}$$

$$U_3(\xi_1, \xi_2, \xi_3) = u_3 + w^i - \xi_3\left[\frac{1}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \varphi_1\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} + \varphi_2\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2}\right] \tag{C3}$$

where $w^i(\xi_1, \xi_2)$ is a known field that describes the initial geometric imperfections in the unloaded state. The functions $F_1(\xi_3)$ and $F_2(\xi_3)$ are user-defined and specifiy the through-the-thickness distributions of the transverse-shear strains. These two functions are required to satisfy $F_1(0) = F_2(0) = 0$ and $F_1'(0) = F_2'(0) = 1$. The nonzero strains at any point of the shell are given by

$$\varepsilon_{11}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[\overset{\circ}{\varepsilon}_{11} + \xi_3\overset{\circ}{\chi}_{11} + F_1(\xi_3)\frac{1}{A_1}\frac{\partial \overset{\circ}{\gamma}_{13}}{\partial \xi_1} - F_2(\xi_3)\frac{\overset{\circ}{\gamma}_{23}}{\rho_{11}}\right] \tag{C4}$$

$$\varepsilon_{22}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[\varepsilon^{\circ}_{22} + \xi_3 \chi^{\circ}_{22} + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{22}} + F_2(\xi_3)\frac{1}{A_2}\frac{\partial \gamma^{\circ}_{23}}{\partial \xi_2}\right] \tag{C5}$$

$$\gamma_{12}(\xi_1, \xi_2, \xi_3) = \frac{\frac{1}{2}\gamma^{\circ}_{12}\left[\left(1 + \frac{\xi_3}{R_1}\right) + \left(1 + \frac{\xi_3}{R_2}\right) + \frac{1}{2}\left(\frac{\xi_3}{R_2} - \frac{\xi_3}{R_1}\right)^2\right] + \xi_3 \chi^{\circ}_{12}\left[\left(1 + \frac{\xi_3}{R_1}\right) + \left(1 + \frac{\xi_3}{R_2}\right)\right]}{\left(1 + \frac{\xi_3}{R_1}\right)\left(1 + \frac{\xi_3}{R_2}\right)}$$
$$+ \frac{F_1(\xi_3)\frac{1}{A_2}\frac{\partial \gamma^{\circ}_{13}}{\partial \xi_2} - F_2(\xi_3)\frac{\gamma^{\circ}_{23}}{\rho_{22}}}{\left(1 + \frac{\xi_3}{R_2}\right)} + \frac{F_2(\xi_3)\frac{1}{A_1}\frac{\partial \gamma^{\circ}_{23}}{\partial \xi_1} + F_1(\xi_3)\frac{\gamma^{\circ}_{13}}{\rho_{11}}}{\left(1 + \frac{\xi_3}{R_1}\right)} \tag{C6}$$

$$\gamma_{13}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_1}\right)}\left[F_1'(\xi_3)\left(1 + \frac{\xi_3}{R_1}\right) - \frac{F_1(\xi_3)}{R_1}\right]\gamma^{\circ}_{13} \tag{C7}$$

$$\gamma_{23}(\xi_1, \xi_2, \xi_3) = \frac{1}{\left(1 + \frac{\xi_3}{R_2}\right)}\left[F_2'(\xi_3)\left(1 + \frac{\xi_3}{R_2}\right) - \frac{F_2(\xi_3)}{R_2}\right]\gamma^{\circ}_{23} \tag{C8}$$

where the reference-surface membrane strains are given by

$$\varepsilon^{\circ}_{11}(\xi_1, \xi_2) = e^{\circ}_{11} + \frac{1}{2}\left(\varphi_1^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e^{\circ}_{11}\right)^2 + e^{\circ}_{12}\left(e^{\circ}_{12} + 2\varphi\right)\right] + c_1 e^{\circ}_{11}\frac{w^i}{R_1} - \varphi_1\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} \tag{C9}$$

$$\varepsilon^{\circ}_{22}(\xi_1, \xi_2) = e^{\circ}_{22} + \frac{1}{2}\left(\varphi_2^2 + c_2\varphi^2\right) + \frac{1}{2}c_1\left[\left(e^{\circ}_{22}\right)^2 + e^{\circ}_{12}\left(e^{\circ}_{12} - 2\varphi\right)\right] + c_1 e^{\circ}_{22}\frac{w^i}{R_2} - \varphi_2\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} \tag{C10}$$

$$\gamma^{\circ}_{12}(\xi_1, \xi_2) = 2e^{\circ}_{12} + \varphi_1\varphi_2 + c_1\left[e^{\circ}_{11}\left(e^{\circ}_{12} - \varphi\right) + e^{\circ}_{22}\left(e^{\circ}_{12} + \varphi\right)\right]$$
$$+ \frac{c_1 w^i}{R_1}\left(e^{\circ}_{12} - \varphi\right) + \frac{c_1 w^i}{R_2}\left(e^{\circ}_{12} + \varphi\right) - \varphi_1\frac{1}{A_2}\frac{\partial w^i}{\partial \xi_2} - \varphi_2\frac{1}{A_1}\frac{\partial w^i}{\partial \xi_1} \tag{C11}$$

with the linear deformation parameters

$$e^{o}_{11}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial u_1}{\partial \xi_1} - \frac{u_2}{\rho_{11}} + \frac{u_3}{R_1} \tag{C12}$$

$$e^{o}_{22}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial u_2}{\partial \xi_2} + \frac{u_1}{\rho_{22}} + \frac{u_3}{R_2} \tag{C13}$$

$$2e^{o}_{12}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} + \frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1} + \frac{u_1}{\rho_{11}} - \frac{u_2}{\rho_{22}} \tag{C14}$$

$$\chi^{o}_{11}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial \varphi_1}{\partial \xi_1} - \frac{\varphi_2}{\rho_{11}} \tag{C15}$$

$$\chi^{o}_{22}(\xi_1, \xi_2) = \frac{1}{A_2}\frac{\partial \varphi_2}{\partial \xi_2} + \frac{\varphi_1}{\rho_{22}} \tag{C16}$$

$$2\chi^{o}_{12}(\xi_1, \xi_2) = \frac{1}{A_1}\frac{\partial \varphi_2}{\partial \xi_1} + \frac{\varphi_1}{\rho_{11}} + \frac{1}{A_2}\frac{\partial \varphi_1}{\partial \xi_2} - \frac{\varphi_2}{\rho_{22}} - \varphi\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{C17}$$

and the linear rotation parameters

$$\varphi_1(\xi_1, \xi_2) = \frac{c_3 u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial \xi_1} \tag{C18}$$

$$\varphi_2(\xi_1, \xi_2) = \frac{c_3 u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial \xi_2} \tag{C19}$$

$$\varphi(\xi_1, \xi_2) = \frac{1}{2}c_3\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial u_1}{\partial \xi_2} + \frac{u_1}{\rho_{11}} + \frac{u_2}{\rho_{22}}\right) \tag{C20}$$

Equilibrium Equations and Boundary Conditions

The equilibrium equations are given by

$$\begin{aligned} &\frac{1}{A_1}\frac{\partial \mathcal{N}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{12}}{\partial \xi_2} - \frac{2\mathcal{N}_{12}}{\rho_{11}} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{22}} + \frac{c_3\tilde{Q}_{13}}{R_1} \\ &\qquad + \frac{c_3}{2A_2}\frac{\partial}{\partial \xi_2}\left[\mathcal{M}_{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right] + \mathcal{P}_1 + q_1 + \mathcal{P}^{i}_1 + q^{i}_1 = 0 \end{aligned} \tag{C21}$$

$$\frac{1}{A_1}\frac{\partial \mathcal{N}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{N}_{22}}{\partial \xi_2} + \frac{\mathcal{N}_{11} - \mathcal{N}_{22}}{\rho_{11}} + \frac{2\mathcal{N}_{12}}{\rho_{22}} + \frac{c_3 \tilde{Q}_{23}}{R_2} + \frac{c_3}{2A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{M}_{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\right] + \mathcal{P}_2 + q_2 + \mathcal{P}_2^i + q_2^i = 0 \tag{C22}$$

$$\frac{1}{A_1}\frac{\partial \tilde{Q}_{13}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \tilde{Q}_{23}}{\partial \xi_2} + \frac{\tilde{Q}_{13}}{\rho_{22}} - \frac{\tilde{Q}_{23}}{\rho_{11}} - \frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2} + \mathcal{P}_3 + q_3 + \mathcal{P}_3^i + q_3^i = 0 \tag{C23}$$

$$\mathcal{Q}_{13} + \frac{\mathcal{F}_{21}}{\rho_{11}} - \frac{\mathcal{F}_{11}}{\rho_{22}} - \frac{1}{A_1}\frac{\partial \mathcal{F}_{11}}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \mathcal{F}_{21}}{\partial \xi_2} = 0 \tag{C24}$$

$$\mathcal{Q}_{23} + \frac{\mathcal{F}_{22}}{\rho_{11}} - \frac{\mathcal{F}_{12}}{\rho_{22}} - \frac{1}{A_1}\frac{\partial \mathcal{F}_{12}}{\partial \xi_1} - \frac{1}{A_2}\frac{\partial \mathcal{F}_{22}}{\partial \xi_2} = 0 \tag{C25}$$

where

$$\tilde{Q}_{13} = \frac{1}{A_1}\frac{\partial \mathcal{M}_{11}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial \xi_2} + \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{\rho_{22}} - \frac{2\mathcal{M}_{12}}{\rho_{11}} \tag{C26}$$

$$\tilde{Q}_{23} = \frac{1}{A_1}\frac{\partial \mathcal{M}_{12}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{22}}{\partial \xi_2} + \frac{\mathcal{M}_{11} - \mathcal{M}_{22}}{\rho_{11}} + \frac{2\mathcal{M}_{12}}{\rho_{22}} \tag{C27}$$

$$\begin{Bmatrix} q_1^i \\ q_2^i \\ q_3^i \end{Bmatrix} = \begin{Bmatrix} -q_3^L \dfrac{1}{A_1}\dfrac{\partial w^i}{\partial \xi_1} \\ -q_3^L \dfrac{1}{A_2}\dfrac{\partial w^i}{\partial \xi_2} \\ q_3^L\left(\dfrac{w^i}{R_1} + \dfrac{w^i}{R_2}\right) + \dfrac{\partial q_3^L}{\partial \xi_3} w^i \end{Bmatrix} \tag{C28}$$

$$\begin{aligned} \mathcal{P}_1 = &-\frac{c_3}{R_1}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{c_2}{2}\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left[\varphi\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\right] + \frac{c_1}{A_1}\frac{\partial}{\partial \xi_1}\left[\mathcal{N}_{11}e_{11}^o + \mathcal{N}_{12}\left(e_{12}^o - \varphi\right)\right] \\ &- \frac{c_1}{\rho_{11}}\left[\mathcal{N}_{11}\left(e_{12}^o + \varphi\right) + \mathcal{N}_{22}\left(e_{12}^o - \varphi\right) + \mathcal{N}_{12}\left(e_{11}^o + e_{22}^o\right)\right] - \frac{c_1}{\rho_{22}}\left[\mathcal{N}_{22}e_{22}^o - \mathcal{N}_{11}e_{11}^o + 2\mathcal{N}_{12}\varphi\right] \\ &+ \frac{c_1}{2}\frac{1}{A_2}\frac{\partial}{\partial \xi_2}\left[\left(\mathcal{N}_{11} - \mathcal{N}_{22}\right)\varphi + 2\left(\mathcal{N}_{22}e_{12}^o + \mathcal{N}_{12}e_{11}^o\right)\right] \end{aligned} \tag{C29}$$

$$\mathcal{P}_2 = -\frac{c_3}{R_2}\left[\mathcal{N}_{22}\varphi_2 + \mathcal{N}_{12}\varphi_1\right] - \frac{c_2}{2A_1}\frac{\partial}{\partial\xi_1}\left[\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi\right] + \frac{c_1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{22}e^{o}_{22} + \mathcal{N}_{12}\left(e^{o}_{12} + \varphi\right)\right]$$
$$+ \frac{c_1}{\rho_{22}}\left[\mathcal{N}_{11}\left(e^{o}_{12} + \varphi\right) + \mathcal{N}_{22}\left(e^{o}_{12} - \varphi\right) + \mathcal{N}_{12}\left(e^{o}_{11} + e^{o}_{22}\right)\right] - \frac{c_1}{\rho_{11}}\left[\mathcal{N}_{22}e^{o}_{22} - \mathcal{N}_{11}e^{o}_{11} + 2\mathcal{N}_{12}\varphi\right] \quad \text{(C30)}$$
$$+ \frac{c_1}{2A_1}\frac{\partial}{\partial\xi_1}\left[\left(\mathcal{N}_{11} - \mathcal{N}_{22}\right)\varphi + 2\left(\mathcal{N}_{11}e^{o}_{12} + \mathcal{N}_{12}e^{o}_{22}\right)\right]$$

$$\mathcal{P}_3 = -\frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{1}{\rho_{22}}\left[\mathcal{N}_{11}\varphi_1 + \mathcal{N}_{12}\varphi_2\right] - \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{12}\varphi_1 + \mathcal{N}_{22}\varphi_2\right]$$
$$+ \frac{1}{\rho_{11}}\left[\mathcal{N}_{12}\varphi_1 + \mathcal{N}_{22}\varphi_2\right] - \frac{c_1}{R_1}\left[\mathcal{N}_{11}e^{o}_{11} + \mathcal{N}_{12}\left(e^{o}_{12} - \varphi\right)\right] - \frac{c_1}{R_2}\left[\mathcal{N}_{22}e^{o}_{22} + \mathcal{N}_{12}\left(e^{o}_{12} + \varphi\right)\right] \quad \text{(C31)}$$

$$\mathcal{P}^{i}_1 = \frac{c_3}{R_1}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right] + \frac{c_1}{A_1}\frac{\partial}{\partial\xi_1}\left[\mathcal{N}_{11}\frac{w^{i}}{R_1}\right]$$
$$+ \frac{c_1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{12}\frac{w^{i}}{R_1}\right] + \frac{c_1 w^{i}}{\rho_{22}}\left[\frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2}\right] - \frac{c_1 w^{i}}{\rho_{11}}\mathcal{N}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad \text{(C32)}$$

$$\mathcal{P}^{i}_2 = \frac{c_3}{R_2}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right] + \frac{c_1}{A_1}\frac{\partial}{\partial\xi_1}\left[\mathcal{N}_{12}\frac{w^{i}}{R_2}\right]$$
$$+ \frac{c_1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{22}\frac{w^{i}}{R_2}\right] + \frac{c_1 w^{i}}{\rho_{11}}\left(\frac{\mathcal{N}_{11}}{R_1} - \frac{\mathcal{N}_{22}}{R_2}\right) + \frac{c_1 w^{i}}{\rho_{22}}\mathcal{N}_{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad \text{(C33)}$$

$$\mathcal{P}^{i}_3 = \frac{1}{A_1}\frac{\partial}{\partial\xi_1}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right] + \frac{1}{A_2}\frac{\partial}{\partial\xi_2}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right]$$
$$+ \frac{1}{\rho_{22}}\left[\mathcal{N}_{11}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{12}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right] - \frac{1}{\rho_{11}}\left[\mathcal{N}_{12}\frac{1}{A_1}\frac{\partial w^{i}}{\partial\xi_1} + \mathcal{N}_{22}\frac{1}{A_2}\frac{\partial w^{i}}{\partial\xi_2}\right] - c_1\left[\frac{\mathcal{N}_{22}}{R_2}\frac{w^{i}}{R_2} + \frac{\mathcal{N}_{11}}{R_1}\frac{w^{i}}{R_1}\right] \quad \text{(C34)}$$

The boundary conditions for an edge given by ξ_1 = constant are given by

$$\mathcal{N}_{11}\left[1 + c_1\left(e^{o}_{11} + \frac{w^{i}}{R_1}\right)\right] + \mathcal{N}_{12}c_1\left(e^{o}_{12} - \varphi\right) = N_1\left(\xi_2\right) \quad \text{or} \quad u_1 = D_1\left(\xi_2\right) \quad \text{(C35)}$$

$$\mathcal{N}_{12} + \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\mathcal{N}_{11}\left(2e^{o}_{12} + \varphi\right) - \mathcal{N}_{22}\varphi + 2\mathcal{N}_{12}\left(e^{o}_{22} + \frac{w^{i}}{R_2}\right)\right]$$
$$+ \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_2} - \frac{1}{R_1}\right) = S_1(\xi_2) + \frac{M_{12}(\xi_2)}{R_2} \quad \text{or} \quad u_2 = D_2(\xi_2) \qquad \text{(C36)}$$

$$\overline{Q}_{13} + \frac{1}{A_2}\frac{\partial \mathcal{M}_{12}}{\partial \xi_2} - \left[\mathcal{N}_{11}\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1}\right) + \mathcal{N}_{12}\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right)\right] = Q_1(\xi_2) + \frac{dM_{12}(\xi_2)}{d\xi_2} \quad \text{or}$$

$$u_3 = D_3(\xi_2) \qquad \text{(C37)}$$

$$\mathcal{M}_{11} = M_1(\xi_2) \quad \text{or} \quad \varphi_1 = \Phi_1(\xi_2) \qquad \text{(C38)}$$

$$\mathcal{F}_{11} = 0 \quad \text{or} \quad \gamma^{o}_{13} = \Gamma_1(\xi_2) \qquad \text{(C39)}$$

$$\mathcal{F}_{12} = 0 \quad \text{or} \quad \gamma^{o}_{23} = \Gamma_2(\xi_2) \qquad \text{(C40)}$$

and

$$\mathcal{M}_{12} = M_{12}(\xi_2) \quad \text{or} \quad u_3 = D_3(\xi_2) \qquad \text{(C41)}$$

at the corners given by $\xi_2 = a_2$ and $\xi_2 = b_2$. The boundary conditions for an edge given by ξ_2 = constant are given by

$$\mathcal{N}_{12} - \frac{c_2}{2}\left(\mathcal{N}_{11} + \mathcal{N}_{22}\right)\varphi + \frac{c_1}{2}\left[\mathcal{N}_{11}\varphi + \mathcal{N}_{22}\left(2e^{o}_{12} - \varphi\right) + 2\mathcal{N}_{12}\left(e^{o}_{11} + \frac{w^{i}}{R_1}\right)\right]$$
$$+ \mathcal{M}_{12}\frac{c_3}{2}\left(\frac{3}{R_1} - \frac{1}{R_2}\right) = S_2(\xi_1) + \frac{M_{21}(\xi_1)}{R_1} \quad \text{or} \quad u_1 = D_1(\xi_1) \qquad \text{(C42)}$$

$$\mathcal{N}_{22}\left[1 + c_1\left(e^{o}_{22} + \frac{w^{i}}{R_2}\right)\right] + \mathcal{N}_{12}c_1\left(e^{o}_{12} + \varphi\right) = N_2(\xi_1) \quad \text{or} \quad u_2 = D_2(\xi_1) \qquad \text{(C43)}$$

$$\overline{Q}_{23} + \frac{1}{A_1}\frac{\partial \mathcal{M}_{12}}{\partial \xi_1} - \left[\mathcal{N}_{12}\left(\varphi_1 - \frac{1}{A_1}\frac{\partial w^{i}}{\partial \xi_1}\right) + \mathcal{N}_{22}\left(\varphi_2 - \frac{1}{A_2}\frac{\partial w^{i}}{\partial \xi_2}\right)\right] = Q_2(\xi_1) + \frac{dM_{21}(\xi_1)}{d\xi_1} \quad \text{or}$$

$$u_3 = D_3(\xi_1) \qquad \text{(C44)}$$

$$\mathcal{M}_{22} = \mathrm{M}_2(\xi_2) \quad \text{or} \quad \varphi_2 = \Phi_2(\xi_1) \quad = 0 \tag{C45}$$

$$\mathcal{F}_{21} = 0 \quad \text{or} \quad \gamma^{\circ}_{13} = \Gamma_1(\xi_1) \tag{C46}$$

$$\mathcal{F}_{22} = 0 \quad \text{or} \quad \gamma^{\circ}_{23} = \Gamma_2(\xi_1) \tag{C47}$$

and

$$\mathcal{M}_{12} = \mathrm{M}_{21}(\xi_1) \quad \text{or} \quad u_3 = \mathrm{D}_3(\xi_1) \tag{C48}$$

at the corners given by $\xi_1 = a_1$ and $\xi_1 = b_1$.

Stresses and Constitutive Equations

The stresses at any point of the shell are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} - \begin{Bmatrix} \bar{\alpha}_{11} \\ \bar{\alpha}_{22} \\ \bar{\alpha}_{12} \end{Bmatrix} \Theta(\xi_1, \xi_2, \xi_3) \right\} \tag{C49}$$

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{55} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{13} \\ \gamma_{23} \end{Bmatrix} \tag{C50}$$

where

$$\Theta(\xi_1, \xi_2, \xi_3) = \hat{\Theta}(\xi_1, \xi_2) G(\xi_3) \tag{C51}$$

is the known temperature field for material points of the shell. The general constitutive equations for a shell are given by

$$\begin{Bmatrix} \mathcal{N}_{11} \\ \mathcal{N}_{22} \\ \mathcal{N}_{12} \end{Bmatrix} = [\mathbf{C_{00}}] \begin{Bmatrix} \varepsilon^{\circ}_{11} \\ \varepsilon^{\circ}_{22} \\ \gamma^{\circ}_{12} \end{Bmatrix} + [\mathbf{C_{01}}] \begin{Bmatrix} \chi^{\circ}_{11} \\ \chi^{\circ}_{22} \\ 2\chi^{\circ}_{12} \end{Bmatrix} + [\mathbf{C_{02}}] \begin{Bmatrix} \dfrac{1}{A_1} \dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_1} \dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_1} \end{Bmatrix} + [\mathbf{C_{03}}] \begin{Bmatrix} \dfrac{1}{A_2} \dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_2} \\ \dfrac{1}{A_2} \dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_2} \end{Bmatrix} + [\mathbf{C_{04}}] \begin{Bmatrix} \gamma^{\circ}_{13} \\ \gamma^{\circ}_{23} \end{Bmatrix} - \hat{\Theta} \{ \bar{\mathbf{\Theta}}_{\mathbf{0}} \} \tag{C52}$$

$$\begin{Bmatrix} \mathcal{M}_{11} \\ \mathcal{M}_{22} \\ \mathcal{M}_{12} \end{Bmatrix} = [\mathbf{C_{10}}] \begin{Bmatrix} \varepsilon^{\circ}_{11} \\ \varepsilon^{\circ}_{22} \\ \gamma^{\circ}_{12} \end{Bmatrix} + [\mathbf{C_{11}}] \begin{Bmatrix} \chi^{\circ}_{11} \\ \chi^{\circ}_{22} \\ 2\chi^{\circ}_{12} \end{Bmatrix} + [\mathbf{C_{12}}] \begin{Bmatrix} \dfrac{1}{A_1} \dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_1} \\ \dfrac{1}{A_1} \dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_1} \end{Bmatrix} + [\mathbf{C_{13}}] \begin{Bmatrix} \dfrac{1}{A_2} \dfrac{\partial \gamma^{\circ}_{13}}{\partial \xi_2} \\ \dfrac{1}{A_2} \dfrac{\partial \gamma^{\circ}_{23}}{\partial \xi_2} \end{Bmatrix} + [\mathbf{C_{14}}] \begin{Bmatrix} \gamma^{\circ}_{13} \\ \gamma^{\circ}_{23} \end{Bmatrix} - \hat{\Theta} \{ \bar{\mathbf{\Theta}}_{\mathbf{1}} \} \tag{C53}$$

$$\begin{Bmatrix} \mathcal{F}_{11} \\ \mathcal{F}_{12} \end{Bmatrix} = [\mathbf{C}_{20}]\begin{Bmatrix} \varepsilon_{11}^{o} \\ \varepsilon_{22}^{o} \\ \gamma_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{21}]\begin{Bmatrix} \chi_{11}^{o} \\ \chi_{22}^{o} \\ 2\chi_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{22}]\begin{Bmatrix} \frac{1}{A_1}\frac{\partial\gamma_{13}^{o}}{\partial\xi_1} \\ \frac{1}{A_1}\frac{\partial\gamma_{23}^{o}}{\partial\xi_1} \end{Bmatrix} + [\mathbf{C}_{23}]\begin{Bmatrix} \frac{1}{A_2}\frac{\partial\gamma_{13}^{o}}{\partial\xi_2} \\ \frac{1}{A_2}\frac{\partial\gamma_{23}^{o}}{\partial\xi_2} \end{Bmatrix} + [\mathbf{C}_{24}]\begin{Bmatrix} \gamma_{13}^{o} \\ \gamma_{23}^{o} \end{Bmatrix} - \hat{\Theta}\{\overline{\mathbf{\Theta}}_2\} \quad \text{(C54)}$$

$$\begin{Bmatrix} \mathcal{F}_{21} \\ \mathcal{F}_{22} \end{Bmatrix} = [\mathbf{C}_{30}]\begin{Bmatrix} \varepsilon_{11}^{o} \\ \varepsilon_{22}^{o} \\ \gamma_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{31}]\begin{Bmatrix} \chi_{11}^{o} \\ \chi_{22}^{o} \\ 2\chi_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{32}]\begin{Bmatrix} \frac{1}{A_1}\frac{\partial\gamma_{13}^{o}}{\partial\xi_1} \\ \frac{1}{A_1}\frac{\partial\gamma_{23}^{o}}{\partial\xi_1} \end{Bmatrix} + [\mathbf{C}_{33}]\begin{Bmatrix} \frac{1}{A_2}\frac{\partial\gamma_{13}^{o}}{\partial\xi_2} \\ \frac{1}{A_2}\frac{\partial\gamma_{23}^{o}}{\partial\xi_2} \end{Bmatrix} + [\mathbf{C}_{34}]\begin{Bmatrix} \gamma_{13}^{o} \\ \gamma_{23}^{o} \end{Bmatrix} - \hat{\Theta}\{\overline{\mathbf{\Theta}}_3\} \quad \text{(C55)}$$

$$\begin{Bmatrix} \mathcal{Q}_{13} \\ \mathcal{Q}_{23} \end{Bmatrix} = [\mathbf{C}_{40}]\begin{Bmatrix} \varepsilon_{11}^{o} \\ \varepsilon_{22}^{o} \\ \gamma_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{41}]\begin{Bmatrix} \chi_{11}^{o} \\ \chi_{22}^{o} \\ 2\chi_{12}^{o} \end{Bmatrix} + [\mathbf{C}_{42}]\begin{Bmatrix} \frac{1}{A_1}\frac{\partial\gamma_{13}^{o}}{\partial\xi_1} \\ \frac{1}{A_1}\frac{\partial\gamma_{23}^{o}}{\partial\xi_1} \end{Bmatrix} + [\mathbf{C}_{43}]\begin{Bmatrix} \frac{1}{A_2}\frac{\partial\gamma_{13}^{o}}{\partial\xi_2} \\ \frac{1}{A_2}\frac{\partial\gamma_{23}^{o}}{\partial\xi_2} \end{Bmatrix} + [\mathbf{C}_{44}]\begin{Bmatrix} \gamma_{13}^{o} \\ \gamma_{23}^{o} \end{Bmatrix} - \hat{\Theta}\{\overline{\mathbf{\Theta}}_4\} \quad \text{(C56)}$$

where the matrices $[\mathbf{C}_{ij}]$ appearing in these equations are given by equations (89) and the vectors $\{\overline{\mathbf{\Theta}}_j\}$ are given by equation (92). Moreover, $[\mathbf{C}_{ji}] = [\mathbf{C}_{ij}]^{\mathbf{T}}$. Specialized forms of $[\mathbf{C}_{ij}]$ and $\{\overline{\mathbf{\Theta}}_j\}$ are given in Appendix B.

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14. ABSTRACT
A detailed exposition on a refined nonlinear shell theory suitable for nonlinear buckling analyses of laminated-composite shell structures is presented. This shell theory includes the classical nonlinear shell theory attributed to Leonard, Sanders, Koiter, and Budiansky as an explicit proper subset. This approach is used in order to leverage the exisiting experience base and to make the theory attractive to industry. In addition, the formalism of general tensors is avoided in order to expose the details needed to fully understand and use the theory. The shell theory is based on "small" strains and "moderate" rotations, and no shell-thinness approximations are used. As a result, the strain-displacement relations are exact within the presumptions of "small" strains and "moderate" rotations. The effects of transverse-shearing deformations are included in the theory by using analyst-defined functions to describe the through-the-thickness distributions of transverse-shearing strains. Constitutive equations for laminated-composite shells are derived without using any shell-thinness approximations, and simplified forms and special cases are presented.

15. SUBJECT TERMS
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