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COMPUTERIZED ANALYSIS OF SHELLS-GOVERNING EQUATIONS

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ABSTRACT

This paper opens with a general discussion of terms in an energy functional which might be the basis from which equations governing stress, stability, and vibration analyses are derived. The energy expression includes strain energy of the shell and discrete stiffeners, kinetic energy of the shell and stiffeners, constraint conditions with Lagrange multipliers, and other terms arising from the change in direction of applied loads during deformation. Brief discussions are included of the coupling effect between bending and extensional energy needed for the analysis of layered composite shells or elastic-plastic shells, nonlinear terms, and the form that the energy expression takes upon discretization of the structure.

A section follows in which the energy formulation for stress, stability, and vibration analyses of an elastic curved beam is given, including thermal effects, moderately large rotations, boundary conditions, and distributed and concentrated loads. The matrix notation and type of discretization are introduced here which will later be used for the analysis of shells of revolution. Terms in the local element stiffness, mass, and load-geometric matrices are derived in terms of nodal point displacements, and it is shown how these local matrices are assembled into global matrices. The purpose of the section is to demonstrate the procedure for derivation of the analogous equations and quantities for shells of revolution or more complex structures.

The next section is on elastic shells of revolution. It opens with a summary of what computer programs exist for stress, buckling, and stability analyses of such structures. The assumptions on which these programs are based are listed and the various components of the energy functional, such as strain energy of the shell and discrete rings, are identified and derived in terms of nodal point displacements. Included are a derivation of the constitutive law for anisotropic shell walls and a formulation of nonlinear constraint conditions, which are required for the treatment of segmented or branched shells with meridional discontinuities between segments or branches. Derivations of terms in the global stiffness and load-geometric matrices and the force vector are given, with tables tracing the origin of each term. The computational strategy for calculation of critical bifurcation buckling loads in the presence of prebuckling nonlinearities is given, with an example of buckling under axial compression of a very thin cylinder. This is a simple problem to formulate but a difficult one to solve numerically, owing to the existence of closely spaced eigenvalues corresponding to nonsymmetric buckling at loads close to the load corresponding to nonlinear axisymmetric collapse. A description of various pitfalls encountered in the search for the lowest bifurcation buckling load is given, including estimates of the critical number of circumferential waves in the buckling mode. Computerized formulations and run times are compared for various discretization methods, including finite difference energy models and standard finite element models, with an example showing comparisons of rate of convergence with increasing nodal point density and computer times required to form stiffness matrices.

Hybrid bodies of revolution are discussed next. By "hybrid" is meant a body of revolution with both one-dimensionally and two-dimensionally discretized regions. The formulation is particularly useful for the stress, buckling, and vibration analyses of branched shells or ring-stiffened shells in which one is particularly interested in local effects within a distance equal to a shell wall thickness of a branch or ring. An appropriate strategy for the solution of nonlinear problems with simultaneous geometric nonlinearity and path-dependent material properties is described, including the development of the incremental constitutive law for the tangent stiffness method of treatment of elastic-plastic structures. The two-dimensionally discretized regions are modeled with use of 8-node isoparametric quadrilaterals of revolution. Details are presented on the formulation of constraint conditions for compatibility at junctions between rotationally symmetric shell segments (one-dimensionally discretized regions) and solid segments (two-dimensionally discretized regions).

The paper closes with a summary of linear equations for general shells. Surface coordinates, the first and second fundamental forms, and the definition of a shell are introduced, and the assumptions corresponding to Love's first approximation are identified. The differences in commonly used or referenced formulations are listed, including differences with regard to kinematic relations, expressions for total strain anywhere in the thickness of the shell wall, and expressions for stress and moment resultants. Comments are offered on which theory is the most suitable for engineering estimates.

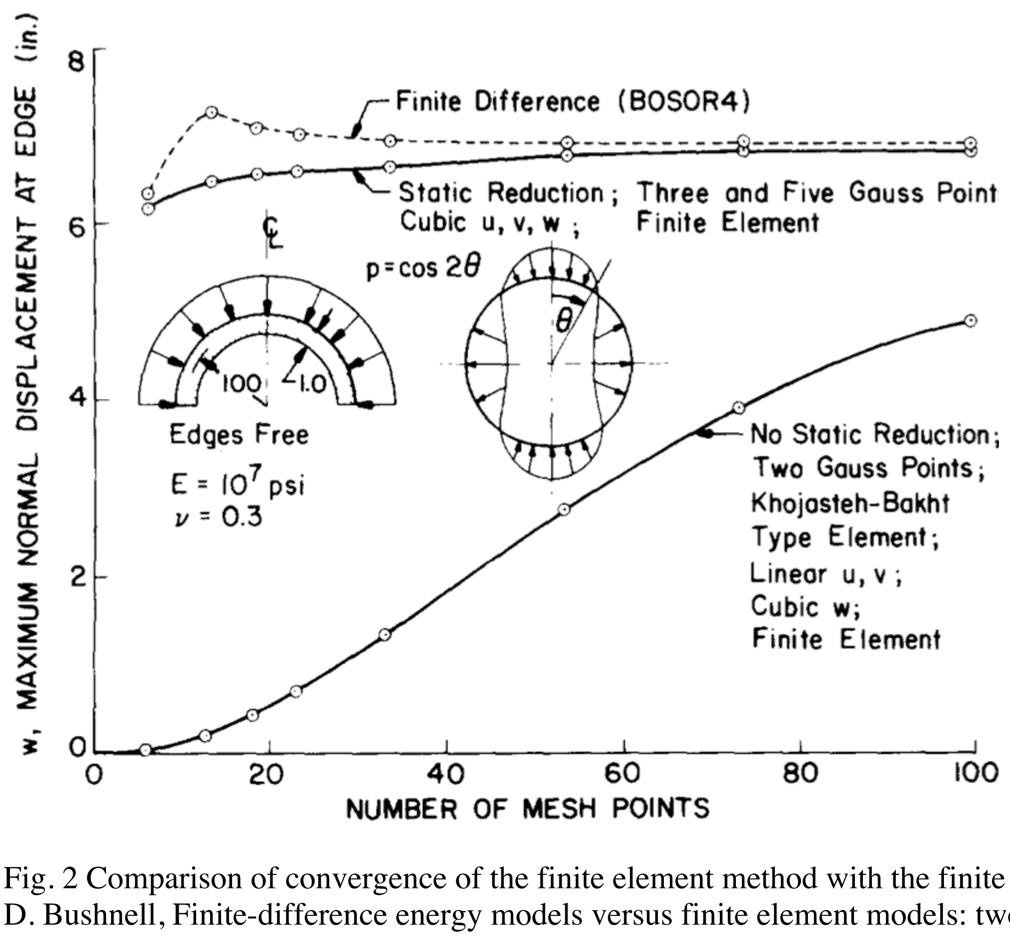

Fig. 2 Comparison of convergence of the finite element method with the finite difference energy method (from D. Bushnell, Finite-difference energy models versus finite element models: two variational approaches in one computer program. *Numerical and Computer Methods in Structural Mechanics* (Edited by S. J. Fenves, N. Perrone, A. R. Robinson and W. C. Schnobrich), pp. 291-336. Academic Press, New York, 1973).

ANALYSIS OF SHELLS OF REVOLUTION

The importance of this class of structures is attested to by the numerous computer programs that have been written for analysis of stress, buckling and vibration of axisymmetric shells.

COMPUTER PROGRAMS

In Fig. 16 the names of computer programs or their authors are located in a space with coordinates that measure complexity of geometry versus generality of phenomenon. Each name indicates the capability of the corresponding computer program to perform the analysis indicated by the intersection of these coordinates. In this coordinate system increasingly general-purpose computer codes lie increasing distances from both axes. Other codes, existing just outside of the region depicted, apply to structures that are "almost" shells of revolution, such as shells with cutouts, shells with material properties that vary around the circumference, or panels of shells of revolution.

The region shown in Fig. 16 is divided by a heavy line into two fields: Programs lying within the heavy line are based on numerical methods that are essentially one-dimensional, that is, the dependent variables are separable and only one spatial variable need be discretized; programs lying outside the heavy line are based on numerical methods in which two or more spatial variables are discretized. It is generally true that analysis methods and programs lying outside the heavy line require perhaps an order of magnitude more computer time for a given case with given nodal point density than do those lying inside the line. This distinction arises because the bandwidths and ranks of equation systems in two-dimensional numerical analyses are greater than those in one-dimensional numerical analyses. Certain of the areas in Fig. 16 are blank. Those near the origin correspond in general to cases for which closed-form solutions exist and for which slightly more general programs are clearly applicable. The blank areas lying near the outer boundaries of the chart are for the most part covered by more general programs such as NASTRAN, SPAR, STAGS, STRUDL, ASKA, MARC, ANSYS and other general-purpose programs described in Ref. [24]. (2011 NOTE: A bit of history!)

As of 1980 the most commonly used computer programs for complex shells of revolution are those by Cohen [l], Kalnins [2], Svalbonas [3], and Bushnell [18,25]. A typical summary of the capabilities of such programs is listed in Table 1. In general the shell-of-revolution codes represent implementation of three distinct analyses:

(1) A nonlinear stress analysis for axisymmetric behavior of axisymmetric shell systems (large deflections, elastic or elastic-plastic).

(2) A linear stress analysis for axisymmetric and non-symmetric behavior of axisymmetric shell systems submitted to axisymmetric and nonsymmetric loads.

(3) An eigenvalue analysis in which the eigenvalues represent buckling loads or vibration frequencies of axisymmetric shell systems submitted to axisymmetric loads. (Eigenvectors may correspond to axisymmetric or nonsymmetric modes.)

Some of the codes [1, 18] have an additional branch corresponding to buckling of nonsymmetrically loaded shells of revolution. In the BOSOR4 program [18] this branch is really a combination of the second and third analyses just listed.

Advantage of axisymmetric geometry: separation of variables

The great advantage of the computer programs cited above is their efficiency. This efficiency derives from the fact that for the three types of analysis just listed the independent variables can be separated and an analytically two-dimensional problem thus reduced to a numerically one-dimensional model. Such a model leads to compact, narrowly banded stiffness, load-geometric, and mass matrices, as we have seen from the beam analysis of the previous section. The reduction of these matrices for solving equilibrium and eigenvalue problems is performed speedily on the computer.

For example, the independent variables of the BOSOR4 analysis [18] are the arc length s measured along the shell reference surface and the circumferential coordinate *theta*. The dependent variables are the displacement components *u, v* and w of the shell wall reference surface. For the three types of analyses listed above it is possible to eliminate the circumferential coordinate *theta* by separation of variables: in the nonlinear stress analysis *theta* is not present; in the linear stress analysis the nonsymmetric load system is expressed as a sum or harmonically varying quantities, the shell response to each harmonic being calculated separately; and in the eigenvalue analysis the eigenvectors vary harmonically around the circumference. Buckling under nonsymmetric loads is handled by calculation of the nonsymmetric prestress distribution from the linear theory and establishment of an eigenvalue problem in which the prestress distribution along a given meridian, presumably the meridian with the most destabilizing prestress, is assumed to be axisymmetric. Thus, the *theta*-dependence, where applicable, is eliminated by the assumption that displacements *u(s,theta)*, *v(s,theta), w(s,theta)* are given by *u(s)sin ntheta, v(s)cos ntheta, w(s)sin ntheta,* or by *u(s)cos ntheta, v(s)sin* n*theta*, *w(s)cos ntheta*.

The advantages of being able to eliminate one of the independent variables cannot be overemphasized. The number of calculations performed by the computer for a given nodal point spacing along the arc length *s* is greatly reduced, leading to significant reductions in computer time. Because the numerical analysis is "one-dimensional" a rather elaborate composite shell structure can be analyzed in a single "pass" through the computer. The disadvantage is, of course, the restriction to axisymmetric (or prismatic) structures.

ENERGY FORMULATION-A SUMMARY

The following analysis of a segmented, ring-stiffened shell of revolution is similar to that for the beam given in the previous section. It is based on energy minimization with constraint conditions. The total energy of the system involves (1) strain energy of the shell segments U_s, (2) strain energy of the discrete rings U_r, (3) potential energy of the applied line loads and pressures U_p, (4) kinetic energy of the shell segments T_s, and (5) kinetic energy of the discrete rings T_r. In addition the total energy functional includes constraint conditions U_c arising from (1) displacement conditions at the ends of the composite shell, and (2) compatibility conditions between adjacent segments of the composite shell. These components of energy and the constraint conditions are initially integrodifferential forms. They are then written in terms of the shell reference surface nodal point displacement components u_i, v_i and w_i and Lagrange multipliers, $Lambda_i$ The integration along the reference surface meridian is performed numerically. Now an algebraic form, the energy is minimized with respect to the discrete dependent variables, u_i, v_i, w_i, and $Lambda_i$.

In the nonlinear stress analysis the energy expression has terms linear, quadratic, cubic, and quartic in the dependent variables. The cubic and quartic terms arise from the "rotation-squared" terms which appear in the constraint conditions and in the kinematic expressions for reference surface strains e_1, e_2, and e_{12}. Nonlinear material properties (plasticity) are not included here. For details on plastic buckling the reader should consult Ref. [26]. Energy minimization leads to a set of nonlinear algebraic equations which are solved by the Newton-Raphson method. Stress and moment resultants are calculated in a straightforward manner from the mesh point displacement components through the constitutive equations (stress-strain law) and kinematic (strain-displacement) relations.

The results from the nonlinear axisymmetric stress analysis are used in the eigenvalue analyses for buckling and vibration. The "prebuckling" or "prestress" meridional and circumferential stress resultants N_{10} and N_{20} and the meridional rotation beta$_0$ appear as known variable coefficients in the energy expression which governs bifurcation buckling and modal vibration. This bifurcation buckling or modal vibration energy expression is a homogenous quadratic form. The values of a parameter (load or frequency) which render the quadratic form stationary with respect to infinitesimal variations of the dependent variables represent buckling loads or natural frequencies. These "eigenvalues" are calculated from a set of linear, homogeneous equations.

Similar linear equations, with a "right-hand-side" vector added, are used for the linear stress analysis of axisymmetrically and nonsymmetrically loaded shells. The "right-hand-side" vector represents load terms and terms due to thermal stress. The variable coefficients N_{10} and N_{20} and beta$_0$ mentioned above are zero, of course, since there is no nonlinear "prestress" phase in the linear nonsymmetric equilibrium analysis.

BASIC ASSUMPTIONS

The assumptions upon which the following analysis is based are:

(1) The wall material is elastic and behaves linearly.

(2) Thin-shell theory holds; i.e. normals to the undeformed surface remain normal and undeformed. Transverse shear deformation is neglected.

(3) The structure is axisymmetric, and in vibration analysis and nonlinear stress analysis the loads and prebuckling or prestress deformations are axisymmetric.

(4) The axisymmetric prebuckling deflections in the nonlinear theory, while considered finite, are moderate; i.e. the square of the meridional rotation can be neglected compared with unity.

(5) In the calculation of displacement and stresses in nonsymmetrically loaded shells, linear theory is used. This analysis is based on standard small-deflection analysis.

(6) A typical cross-section dimension of a discrete ring stiffener is small compared with the radius of the ring.

(7) The cross-sections of the discrete rings remain undeformed as the structure deforms, and the rotation about the ring centroid is equal to the rotation of the shell meridian at the attachment point of the ring.

(8)The discrete ring centroids coincide with their shear centers.

(9) If meridional stiffeners are present, they are numerous enough to include in the analysis by an averaging or 'smearing' of their properties over any parallel circle of the shell structure.

(10) The shell is thin enough to neglect terms of order t/R compared to unity, where t is a typical thickness and R a typical radius of curvature.

(11) Prebuckling in-plane shear resultants are neglected in the stability analysis.

(12) The integrated constitutive law is restricted to the form given in eqn (84). For example, any coupling between normal stress resultants and shearing and twisting motions is neglected.

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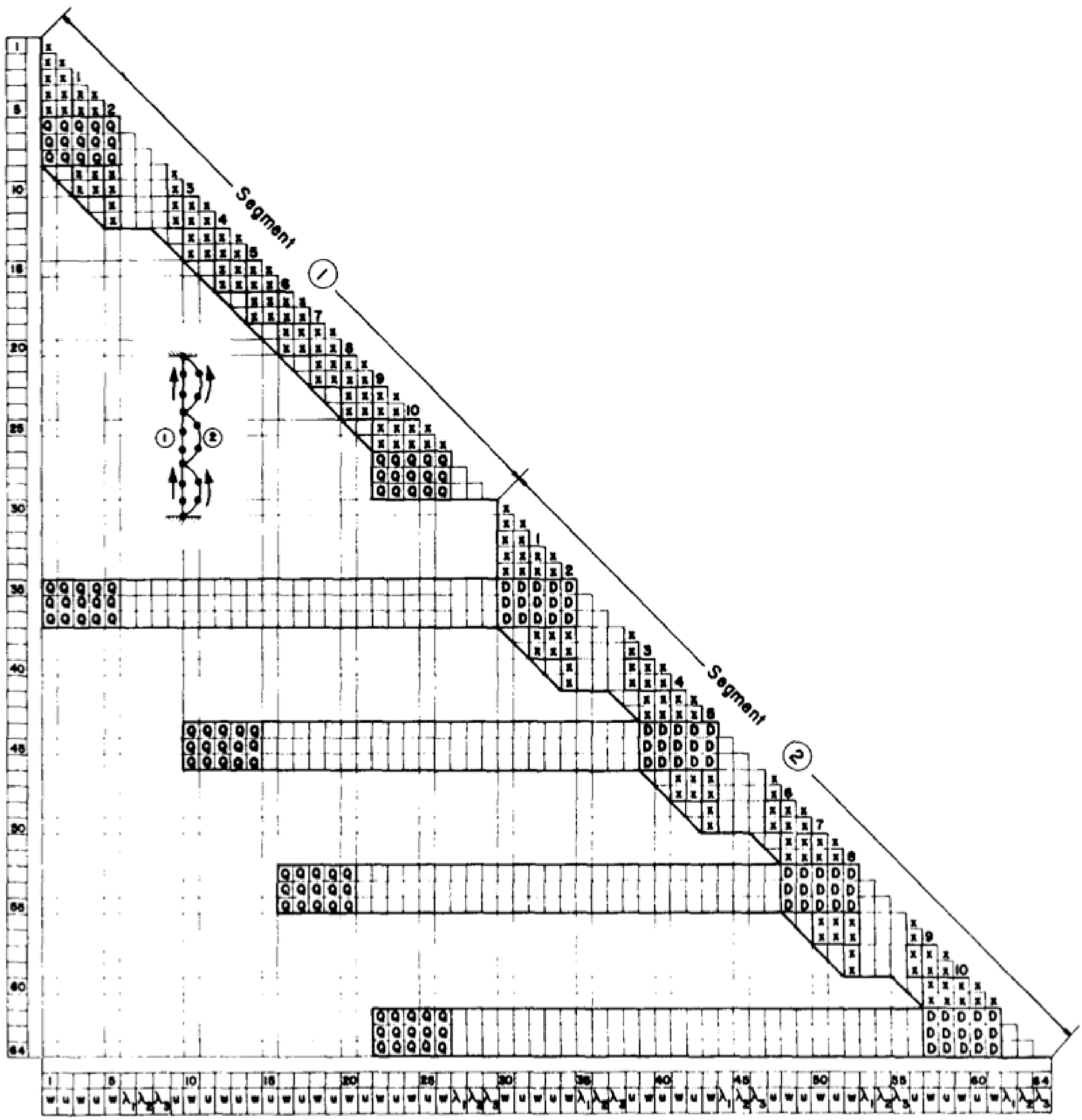

Fig. 13 Stiffness matrix configuration for a two-segment, one-dimensionally discretized structure with intermittent fasteners (from D. Bushnell, Stress, stability, and vibration of complex branched shells of revolution. *Computers & Structures* **4,** 399-435, 1974).